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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Best m-term Approximation and Lizorkin-Triebel Spaces

M. Hansen, W. Sickel

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AG Numerik/Optimierung
Fachbereich 12 - Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Str.
35032 Marburg

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Markus Hansen * & Winfried Sickel
Friedrich-Schiller-University Jena

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Abstract

We shall investigate the asymptotic behaviour of the widths of best m-term approximation with respect to Lizorkin-Triebel as well as Besov spaces. Our approach leads to final assertions in all possible situations. Furthermore, we shall also discuss embeddings into the approximation spaces $\mathcal{A}^s_q(L_p)$ with $q < \infty$. This leads to detailed information on the decay of wavelet coefficients for the elements of Lizorkin-Triebel and Besov spaces.

1 Introduction

Let $\Phi := (\psi_j)_j$ denote a wavelet basis satisfying some additional smoothness, integrability, and moment conditions. We consider best m-term approximation with respect to Φ, i.e., we investigate the quantity

$$\sigma_m(f,\Phi)_X := \inf\left\{ \| f - \sum_{j\in\Lambda} c_j\, \psi_j \|_X : \quad |\Lambda| \le m\,, \quad c_j \in \mathbb{C}\,,\, j \in \Lambda \right\}, \qquad m \in \mathbb{N}_0\,.$$

Associated widths are defined as follows. Let X and Y be quasi-Banach spaces such that $Y \hookrightarrow X$. Then we define

$$\sigma_m(Y,X,\Phi) := \sup\left\{ \sigma_m(f,\Phi)_X : \quad \| f\, |Y\| \le 1 \right\}, \qquad m \in \mathbb{N}_0\,. \qquad (1)$$

Usually one concentrates on $X = L_p(\mathbb{R}^d)$. However, there is some motivation to consider also more general cases.

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Let $s > 0$ and $1 < p < \infty$. Under appropriate extra conditions on the basis Φ one knows that the approximation space $\mathcal{A}^s_\infty(L_p(\mathbb{R}^d), \Phi)$, defined by

$$f \in L_p(\mathbb{R}^d)\, , \qquad \sup_{m=0,1,\ldots} \, (m+1)^s \, \sigma_m(f,\Phi)_{L_p(\mathbb{R}^d)} < \infty \, ,$$

is given by

$$\Big\{ f = \sum_j c_j \, \psi_j \, : \quad (c_j)_j \in \ell_{\tau,\infty} \Big\}\, , \qquad \frac{1}{\tau} := \frac{s}{d} + \frac{1}{p}\, , \tag{2}$$

see [13] or [28], and also [20]. However, a characterization of $\mathcal{A}^s_\infty(L_p(\mathbb{R}^d), \Phi)$ in classical terms of smoothness like derivatives, differences or moduli of smoothness seems to be unknown. Besov and Lizorkin-Triebel spaces are well-known examples of spaces allowing a characterization in such terms. Today Besov spaces are indispensable in approximation theory. Lizorkin-Triebel spaces are certain generalizations of Sobolev spaces. They occur much less in approximation theory, but in particular the classes $F^s_{p,\infty}(\Omega)$, sometimes denoted as $C^s_p(\Omega)$, have played a rôle in connection with the sharp maximal function, see [5], [14] and [42, 1.7.2, 5.3]. Furthermore, there is the famous result of DeVore, Jawerth and Popov [11]. Let $0 < \tau < p$. A function f belongs to the Besov space $B^{d(\frac{1}{\tau}-\frac{1}{p})}_{\tau,\tau}(\mathbb{R}^d)$ if, and only if it belongs to the approximation space $\mathcal{A}^{\frac{1}{\tau}-\frac{1}{p}}_\tau(L_p(\mathbb{R}^d), \Phi)$, i.e., it satisfies

$$\Big(\sum_{m=1}^\infty m^{-1} \, \Big[m^{\frac{1}{\tau}-\frac{1}{p}} \, \sigma_m(f,\Phi)_{L_p(\mathbb{R}^d)} \Big]^\tau \Big)^{1/\tau} < \infty \, . \tag{3}$$

Here in this paper we shall deal with the following more simple problem: we shall characterize all Besov and Lizorkin-Triebel spaces which are embedded into $\mathcal{A}^s_\infty(L_p(\mathbb{R}^d), \Phi)$. In fact, we shall deal with this problem by replacing also $L_p(\mathbb{R}^d)$ by Besov and Lizorkin-Triebel spaces. Such a program has been initiated by Kyriazis [29], but see also [11], [26], [24], [8] and [7] for earlier results in this direction.
We shall divide our investigations into three different cases. In a first case we shall study the behaviour of $\sigma_m(Y, X, \Phi)$ for pairs (X, Y) of homogeneous Besov and Lizorkin-Triebel spaces. Then we continue by investigating inhomogeneous Besov and Lizorkin-Triebel spaces. Finally, we deal with spaces of such type on bounded open sets. All three cases can be handled by essentially the same methods. However, there are some differences in the outcome. Whereas the results in the limiting situations coincide, this turns out to be not the case in the non-limiting situations. By limiting situations we understand all embeddings which are covered by Lemma 1 below. For homogeneous spaces no further embeddings exist. But for inhomogeneous spaces and for spaces on domains the theory of embeddings is much richer, see Lemma 3 and Figure 1 below. In all non-limiting situations it makes an essential

difference whether the underlying domain is bounded or unbounded.

Concerning the wavelet system Φ we wish to remark the following. First of all, we use different systems for homogeneous spaces, for inhomogeneous spaces on $\mathbb{R}^d$ and for spaces on bounded open sets Ω. Exact definitions are given in (32), (26), and (39). When we deal with the widths $\sigma_m(Y, X, \Phi)$ it is always assumed that Y and X allow a characterization by means of the system Φ, see Propositions 1, 2 (homogeneous spaces) and Propositions 3, 4 (inhomogeneous spaces on $\mathbb{R}^d$) for sufficient conditions. In case of spaces on domains we will suppose that the associated inhomogeneous spaces on $\mathbb{R}^d$ allow a characterization by Φ, since these spaces will be defined as restrictions of the inhomogeneous spaces.

In this paper we are not interested in optimal bases or the exact determination of the best m-term approximation, we refer, e.g., to [10, 44, 15, 26, 24] for those aspects of the theory. In our context it always holds, that

$$\sigma_m\big(f, \dot{X}^s_{p,q}(\mathbb{R}^d), \Phi\big) \asymp \inf\Big\{ \big\| f - \sum_{j\in\Lambda} \langle f, \psi_j\rangle\, \psi_j \,\big|\, \dot{X}^s_{p,q}(\mathbb{R}^d)\big\| : \quad |\Lambda| \le m \Big\}\,,$$

$X \in \{B, F\}$. With other words, it will be sufficient to approximate f by appropriate partial sums of the Fourier-wavelet expansion of f.

The paper is organized as follows. In Section 2 we state and comment on our main results. Section 3 contains some supplements dealing with the decay of wavelet coefficients of the elements of $\dot{F}^s_{p,q}$, $\dot{B}^s_{p,q}$ in terms of Lorentz sequence spaces. All proofs will be collected in Section 4. The first step in our proofs will always be the application of a wavelet isomorphism. This reduces the problem for distribution spaces to a problem for sequence spaces. Wavelet isomorphisms will be described in Subsection 4.1. On the level of sequence spaces we use the characterization of the approximation spaces in terms of Lorentz spaces, see e.g. [37], as well as the fact that the approximation spaces with respect to Lizorkin-Triebel spaces do not depend on the fine-index q, see [29]. Our main tool will be interpolation theory. In Subsections 4.4, 4.7 we collect some more material on embeddings of Besov and Lizorkin-Triebel spaces into approximation spaces.

Our methods also work in more complicated situations as tensor products of the spaces considered here or the slightly more general spaces of dominating mixed smoothness. For more details we refer to [22].

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. If X and Y are two quasi-Banach spaces, then the symbol $Y \hookrightarrow X$ indicates that the

embedding is continuous. As usual, the symbol c denotes positive constants which depend only on the fixed parameters s, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols "$\lesssim$" and "$\gtrsim$" instead of "$\leq$" and "$\geq$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq c\,B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For a discrete set ∇ the symbol $|\nabla|$ denotes the cardinality of this set.
General information about homogeneous and inhomogeneous Besov and Lizorkin-Triebel spaces can be found, e.g. in [34, 41, 43] ($B^s_{p,q}$, $F^s_{p,q}$) and [18, 25, 30, 34, 41] ($\dot{B}^s_{p,q}$, $\dot{F}^s_{p,q}$). We will not give definitions here. However, the wavelet characterizations, recalled in Subsection 4.1, can be taken as definitions as well.

2 The asymptotic behaviour of the widths of best m-term approximation

We shall investigate the asymptotic behaviour of the widths $\sigma_m(Y, X, \Phi)$ in three different situations. First we study the case of homogeneous spaces of Besov-Lizorkin-Triebel type. In the second case we turn to inhomogeneous spaces of this type on unbounded domains and finally we consider spaces on bounded domains. Since we need to have $Y \hookrightarrow X$ this puts different restrictions to the admissible pairs (X, Y).

2.1 Widths of best m-term approximation and homogeneous spaces

Here we deal with homogeneous spaces of Besov and Lizorkin-Triebel type.
To begin with, we recall the known embedding relations.

Lemma 1. *Let $s, s_0, s_1 \in \mathbb{R}$ and $0 < q, q_0, q_1 \leq \infty$.*
(i) *Let $0 < p_0, p_1 < \infty$. We have $\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ if and only if*

$$s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1} \tag{4}$$

and either $p_0 < p_1$ or $p_0 = p_1$ and $q_0 \leq q_1$.
(ii) *Let $0 < p_0, p_1 \leq \infty$. We have $\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ if and only if (4), $p_0 \leq p_1$, and $q_0 \leq q_1$ hold.*
(iii) *Let $0 < p_0 < p_1 \leq \infty$ and suppose*

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}\,. \tag{5}$$

We have

$$\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s}_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d)$$

if and only if $q_0 \le p \le q_1$.
(iv) *Let* $0 < p < \infty$. *We have*

$$\dot{B}^{s}_{p,q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s}_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s}_{p,q_1}(\mathbb{R}^d)$$

if and only if $q_0 \le \min(p,q)$ *and* $\max(p,q) \le q_1$.

Remark 1. (i) None of these embeddings is compact.
(ii) For proofs we refer, e.g., to [25] (sufficiency) and to [38] (necessity). In the latter reference only inhomogeneous spaces are considered. However, the arguments carry over. The reference [25] does not cover the second embedding in part (iii). For this part we refer to Franke [17], but see also [45] and [23].

In the list of all possible embeddings only those with $p_0 < p_1$ are of interest in our context.

Lemma 2. *Let* $\dot{Y}^{s}_{p,q_0}(\mathbb{R}^d) \hookrightarrow \dot{X}^{s}_{p,q_1}(\mathbb{R}^d)$ *be one of the possible embeddings in Lemma 1. Then*

$$\sigma_m\Big(\dot{Y}^{s}_{p,q_0}(\mathbb{R}^d), \dot{X}^{s}_{p,q_1}(\mathbb{R}^d), \Phi\Big) \asymp 1\,, \qquad m \in \mathbb{N}\,. \tag{6}$$

From now on, we concentrate on $p_0 < p_1$. To begin with, we first consider the case where both, X and Y, are spaces of Lizorkin-Triebel type.

Theorem 1. *Let* $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \le \infty$ *and* s_0, s_1 *as in (4). Then*

$$\sigma_m\Big(\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^d), \dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{p_1}}\,, \qquad m \in \mathbb{N}\,. \tag{7}$$

Remark 2. (i) Since the embedding $\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ is not compact other widths of this embedding like approximation numbers, Kolmogorov numbers or entropy numbers would not tend to 0 for $m \to \infty$.
(ii) The behaviour of σ_m does not depend on q_0, q_1. This is in sharp contrast with the other cases treated below.

Theorem 2. *Let* $0 < p_0 < p_1 < \infty$, $0 < q_1 \le \infty$, $0 < q_0 \le p_1$, *and* s_0, s_1 *as in (4). We put* $t := \max(p_0, q_0)$. *Then*

$$\sigma_m\Big(\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^d), \dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{t}+\frac{1}{p_1}}\,, \qquad m \in \mathbb{N}\,. \tag{8}$$

Remark 3. (i) In view of Lemma 1(iii) the restriction $0 < q_0 \le p_1$ is necessary.
(ii) This time the behaviour of σ_m does not depend on q_1. In both cases, in Thm. 1 and Thm. 2, this expresses the fact that the approximation spaces $\mathcal{A}^s_\infty(\dot F^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi)$ do not depend on q_1 (at least under appropriate restrictions with respect to Φ). This phenomenon has been observed for the first time by Kyriazis [29].

Theorem 3. *Let $0 < p_0 < p_1 \le \infty$, $0 < q_0 \le \infty$, $p_0 \le q_1 \le \infty$, and s_0, s_1 as in (4). We put $t := \min(p_1, q_1)$. Then*

$$\sigma_m\Big(\dot F^{s_0}_{p_0,q_0}(\mathbb{R}^d), \dot B^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{t}}\,, \qquad m \in \mathbb{N}\,. \tag{9}$$

Remark 4. The restriction $p_0 \le q_1 \le \infty$ is necessary in this context, see Lemma 1(iii).

Finally we turn to the embedding of Besov into Besov spaces.

Theorem 4. *Let $0 < p_0 < p_1 \le \infty$, $0 < q_0 \le q_1 \le \infty$, and s_0, s_1 as in (4). We put*

$$r := \min\Big(\frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{q_0} - \frac{1}{q_1}\Big)$$

Then

$$\sigma_m\Big(\dot B^{s_0}_{p_0,q_0}(\mathbb{R}^d), \dot B^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-r}\,, \qquad m \in \mathbb{N}\,. \tag{10}$$

Remark 5. (i) The restriction $q_0 \le q_1$ is necessary in this context, see Lemma 1(ii).
(ii) The limiting case for the estimate from above in (10), i.e. the case $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{q_0} - \frac{1}{q_1}$, has been already investigated by Kyriazis [29]. The non-limiting cases can be easily traced back to the limiting case.
(iii) Summarizing, we have determined the asymptotic behaviour of $\sigma_m\Big(\dot Y^{s_0}_{p_0,q_0}(\mathbb{R}^d), \dot X^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big)$ in all reasonable situations. The picture is complete. However, by definition we have excluded the very interesting scale $\dot F^{s_0}_{\infty,q_0}(\mathbb{R}^d)$ which includes BMO. Best m-term approximation in BMO, VMO or more generally in $\dot F^{s_0}_{\infty,q_0}(\mathbb{R}^d)$ is quite different from what we are doing here. In our context that part of the wavelet expansion with the m largest coefficients (by modulus) yields a nearly optimal approximation. This is wrong with respect to the scale $\dot F^{s_0}_{\infty,q_0}(\mathbb{R}^d)$, see [26] and [32].

2.2 Best m-term approximation in L_p-norms

Restricted to this subsection, we shall use the more precise notation $[f]$ instead of f, see (25). Furthermore, by W we denote the mapping

$$[f] \mapsto (\langle [f], \psi^e_{j,k}\rangle)_{j,k,e}\,.$$

Let $1 < p < \infty$. Then it is well-known, see [35], [39] or [21], that there exists a linear isomorphism $T : \dot{F}^0_{p,2}(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ such that the equivalence class $[T([f])]$ of the image of $[f] \in \dot{F}^0_{p,2}(\mathbb{R}^d)$ coincides with $[f]$. More precisely, in each class $[f] \in \dot{F}^0_{p,2}(\mathbb{R}^d)$ there is exactly one element $T([f])$ which belongs to $L_p(\mathbb{R}^d)$ and $\| T([f]) \,| L_p(\mathbb{R}^d)\| \asymp \| [f] \,| \dot{F}^0_{p,2}(\mathbb{R}^d)\|$. This opens the door for an easy interpretation of Theorems 1 and 2. For given m and $[f]$ let

$$G_m([f]) := \sum_{(j,k,e)\in\Lambda_m} \langle [f], \psi^e_{j,k}\rangle \, \psi^e_{j,k} \,, \qquad \Lambda_m \subset \mathbb{Z}\times\mathbb{Z}^d \times E, \quad |\Lambda_m| \le m \,,$$

be chosen in such a way that (cf. remark 14)

$$\| W([f]) - W([G_m([f])]) \,| \dot{f}^0_{p_1,2}\| = \sigma_m(W([f]), \mathcal{B})_{\dot{f}^0_{p_1,2}} \,.$$

Of course, $T([G_m([f])]) = G_m([f])$ for all m since the elements of our wavelet system Φ belong to $L_{p_1}(\mathbb{R}^d)$. As Φ is an unconditional basis in $\dot{F}^0_{p_1,2}(\mathbb{R}^d)$ it holds $\lim_{m\to\infty} \| [f] - [G_m([f])] \,| \dot{F}^0_{p_1,2}(\mathbb{R}^d)\| = 0$. Thus $(G_m([f]))_m$ is a Cauchy sequence in $L_{p_1}(\mathbb{R}^d)$ as well and its limit is obviously given by the wavelet expansion of $[f]$. In this sense we define

$$\sigma_m([f],\Phi)_{L_{p_1}(\mathbb{R}^d)} := \tag{11}$$
$$\inf \Big\{ \Big\| \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E} \langle [f], \psi^e_{j,k}\rangle \, \psi^e_{j,k} - \sum_{(j,k,e)\in\Lambda_m} c^e_{j,k} \, \psi^e_{j,k} \Big| L_{p_1}(\mathbb{R}^d) \Big\| : \quad \Lambda_m \subset \mathbb{Z}\times\mathbb{Z}^d\times E, \quad |\Lambda_m| \le m, \quad c^e_{j,k} \in \mathbb{C} \Big\}$$

and accordingly

$$\sigma_m\Big(\dot{Y}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\mathbb{R}^d), L_{p_1}(\mathbb{R}^d), \Phi\Big) := \sup \Big\{ \sigma_m([f],\Phi)_{L_{p_1}(\mathbb{R}^d)} : \quad \| [f] \,| \dot{Y}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\mathbb{R}^d)\| \le 1 \Big\} .$$

Now we can interpret Theorems 1, 2 as follows.

Corollary 1. *Let $0 < p_0 < p_1 < \infty$ and $1 < p_1 < \infty$.*
(i) *Let $0 < q_0 \le \infty$. Then*

$$\sigma_m\Big(\dot{F}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\mathbb{R}^d), L_{p_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{p_1}} \,, \qquad m \in \mathbb{N} . \tag{12}$$

(ii) *Let $0 < q_0 < p_1$. We put $t := \max(p_0, q_0)$. Then*

$$\sigma_m\Big(\dot{B}^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_{p_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{t}+\frac{1}{p_1}} \,, \qquad m \in \mathbb{N} . \tag{13}$$

Now we turn to the case $p_1 = 1$. First we recall the corresponding embedding assertions. Let $0 < p_0 < 1$. Then

$$\dot{B}^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$$

if and only if $0 < q_0 \le 1$ and

$$\dot{F}^{d(\frac{1}{p_0}-1)}_{p_0,\infty}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)\,,$$

see e.g. [25] and [38] for some details. These embeddings have to be interpreted in the sense that there exists a linear continuous injection $T : \dot{Y}^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d) \to L_1(\mathbb{R}^d)$. This will be sufficient to carry over the arguments from above and to use (11) also with $p_1 = 1$.

Corollary 2. *Let* $0 < p_0 < 1$.
(i) *Let* $0 < q_0 \le \infty$. *Then*

$$\sigma_m\Big(\dot{F}^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_1(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+1}\,, \qquad m \in \mathbb{N}\,. \tag{14}$$

(ii) *Let* $0 < q_0 < 1$. *If* $\frac{1}{p_0} - 1 \le \frac{1}{q_0}$, *then*

$$\sigma_m\Big(\dot{B}^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_1(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+1}\,, \qquad m \in \mathbb{N}\,. \tag{15}$$

2.3 An extremal property of $\dot{F}^s_{p,\infty}$

Let $\mathcal{D}$ be a subset of the quasi-Banach space X. Then we define

$$\sigma_m(a, \mathcal{D})_X := \inf\Big\{ \|\, a - \sum_{j\in\Lambda} c_j\, \psi_j \,\|_X : \quad |\Lambda| \le m\,, \quad c_j \in \mathbb{C}\,,\ \psi_j \in \mathcal{D}\,, j \in \Lambda \Big\}\,,$$

Obviously $\sigma_0(a, \mathcal{D})_X = \|\, a\,\|_X$. We are interested in the approximation spaces relative to σ_m. Let $s > 0$. We define $\mathcal{A}^s_q(X, \mathcal{D})$ to be the collection of all elements a of X such that

$$\|\, a\,|\mathcal{A}^s_q(X,\mathcal{D})\| := \begin{cases} \Big(\sum\limits_{m=0}^{\infty} (m+1)^{-1}\Big[(m+1)^s\,\sigma_m(f,\mathcal{D})_X\Big]^q\Big)^{1/q} & \text{if} \quad 0 < q < \infty\,, \\ \sup\limits_{m=0,1,\ldots} (m+1)^s\,\sigma_m(f,\mathcal{D})_X & \text{if} \quad q = \infty\,, \end{cases}$$

is finite, see e.g. [36, 37, 28, 29].

Remark 6. For later use we mention the continuous embedding $\mathcal{A}^s_{q_0}(X,\mathcal{D}) \hookrightarrow \mathcal{A}^s_{q_1}(X,\mathcal{D})$, $q_0 < q_1$, which becomes obvious by switching to dyadic subsequences.

Now we turn to the specific situation $X := L_{p_1}(\mathbb{R}^d)$ and ask for the largest space $\dot{Y}^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ within the scales of homogeneous Besov and Lizorkin-Triebel spaces such that

$$\dot{Y}^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_\infty(L_{p_1}(\mathbb{R}^d), \Phi) \tag{16}$$

for a given $s > 0$. By means of the results of the previous subsection its easy to determine this space.

Theorem 5. *Let $1 < p_1 < \infty$ and let $s > 0$. Let $\dot{Y}(\mathbb{R}^d)$ be a homogeneous Besov or Lizorkin-Triebel space such that $\dot{Y}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_\infty(L_{p_1}(\mathbb{R}^d), \Phi)$. Then $\dot{Y}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s_0}_{p_0,\infty}(\mathbb{R}^d)$ follows, where*

$$s_0 := d\left(\frac{1}{p_0} - \frac{1}{p_1}\right) \qquad \text{and} \qquad \frac{1}{p_0} := s + \frac{1}{p_1}\,. \tag{17}$$

Remark 7. Theorem 5 remains true for the inhomogeneous spaces on $\mathbb{R}^d$ and as well for the spaces on bounded domains.

2.4 Widths of best m-term approximation and inhomogeneous spaces on $\mathbb{R}^d$

The theory of embeddings of inhomogeneous spaces is much richer than in case of homogeneous spaces.

Lemma 3. *Let $s, s_0, s_1 \in \mathbb{R}$ and $0 < q, q_0, q_1 \le \infty$.*
(i) *All embeddings collected in Lemma 1 have their inhomogeneous counterparts.*
(ii) *Let $0 < p_0 \le p_1 \le \infty$ and suppose*

$$s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}\,. \tag{18}$$

Then $Y^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow X^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ holds with $Y, X \in \{F, B\}$.

Remark 8. (i) Again all these embeddings are not compact.
(ii) In Figure 1 below we have plotted the situation with $s_1 = 0$ and by ignoring the influence of q_0 and q_1. Only the line connecting the pairs $(0, 1/p_1)$ and $(s_0, 1/p_0)$ refers to embeddings in the homogeneous situation, see Lemma 1. We shall call it limiting situation in the inhomogeneous case.
(iii) For proofs (necessity) and further references, we refer to [38].

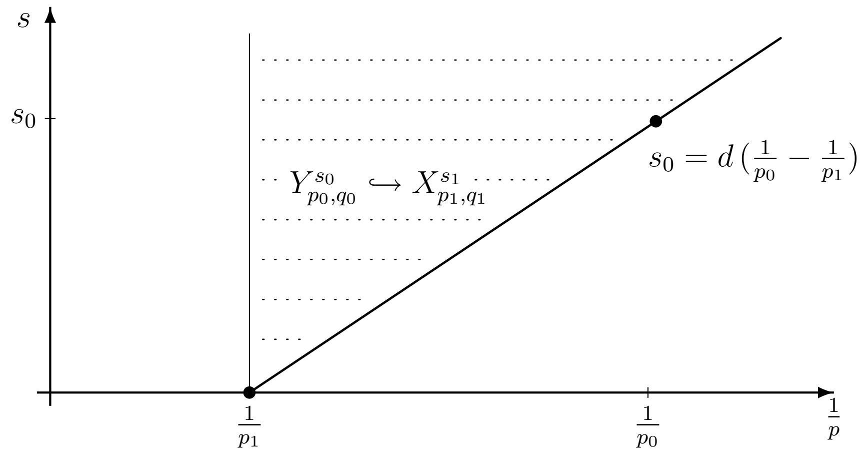

Fig. 1

Theorem 6. (i) *Mutatis mutandis, Theorem 1 - Theorem 5, Corollary 1 and Corollary 2(i) remain true by replacing the spaces with dot by the spaces without a dot.*
(ii) *Let* $0 < q_0 < 1$ *and* $t = \max(p_0, q_0)$. *Then*

$$\sigma_m\Big(B^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_1(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{t}+1}\,, \qquad m \in \mathbb{N}\,. \tag{19}$$

(iii) *Under the restrictions of Lemma 3(ii) we have*

$$\sigma_m\Big(Y^{s_0}_{p_0,q_0}(\mathbb{R}^d), X^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{p_1}}\,, \qquad m \in \mathbb{N}\,. \tag{20}$$

Remark 9. For the non-limiting situation, i.e. if

$$p_0 \le p_1 \qquad \text{and} \qquad s_0 - s_1 > d\left(\frac{1}{p_0} - \frac{1}{p_1}\right),$$

we wish to mention that the behaviour of $\sigma_m\Big(Y^{s_0}_{p_0,q_0}(\mathbb{R}^d), X^{s_1}_{p_1,q_1}(\mathbb{R}^d), \Phi\Big)$ does not depend on the relation of s_0 to s_1. This is in sharp contrast to the situation on bounded domains, see the next subsection.

2.5 Widths of best m-term approximation and spaces on bounded open sets

Let Ω be a bounded open subset of $\mathbb{R}^d$. Then all embeddings from Lemma 3 remain true in this situation. But now we also have $X^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow X^{s_0}_{p_1,q_0}(\Omega)$ if $p_0 > p_1$. Moreover, it is known that under the restriction

$$s_0 - s_1 > d\left(\frac{1}{p_0} - \frac{d}{p_1}\right)_+ \tag{21}$$

the embedding $X^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow Y^{s_1}_{p_1,q_1}(\Omega)$ is compact, see [43, Thm. 4.33]. Here $Y, X \in \{B, F\}$.

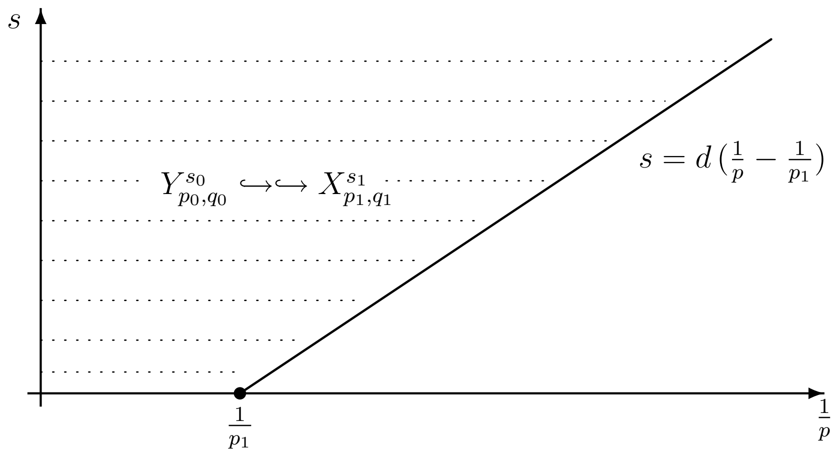

Fig. 2

Now we turn to the asymptotic behaviour of σ_m for those compact embeddings.

Theorem 7. *Let Ω be a bounded open set in $\mathbb{R}^d$. Let $0 < p_0, p_1, q_0, q_1 \le \infty$.*
(i) *Mutatis mutandis, Theorem 1 - Theorem 5, Corollary 1 and Corollary 2 remain true by replacing the spaces with dot by the spaces on Ω.*
(ii) *Let s_0 and s_1 be as in (21). Then, with $Y, X \in \{B, F\}$ ($Y = B$ if $p_0 = \infty$ and $X = B$ if $p_1 = \infty$), we have*

$$\sigma_m\Big(Y^{s_0}_{p_0,q_0}(\Omega), X^{s_1}_{p_1,q_1}(\Omega), \Phi\Big) \asymp m^{-(s_0-s_1)/d}\,, \qquad m \in \mathbb{N}\,. \tag{22}$$

Remark 10. (i) Since the embeddings in Theorem 7(ii) are compact we can compare the behaviour of the widths of best m-term approximation with other widths. Most simple seems to be the comparison with entropy numbers. Under the given restrictions in Theorem 7 (ii) we have

$$e_m\Big(\mathrm{id}\,, Y^{s_0}_{p_0,q_0}(\Omega), X^{s_1}_{p_1,q_1}(\Omega)\Big) \asymp \sigma_m\Big(Y^{s_0}_{p_0,q_0}(\Omega), X^{s_1}_{p_1,q_1}(\Omega), \Phi\Big)$$

where e_m denotes the m-th dyadic entropy number of the identity $\mathrm{id}\, : Y^{s_0}_{p_0,q_0}(\Omega) \to X^{s_1}_{p_1,q_1}(\Omega)$, cf., e.g., [16, Thm. 3.3.2].
(ii) Theorem 7(ii) is essentially proved in [9].

3 Decay of wavelet coefficients

There is a close relationship between embeddings into approximation spaces with respect to best m-term approximation and decay of the wavelet coefficients. This is expressed by the following theorem.

Theorem 8. *Suppose* $0 < p_0 < \infty$ *and* $0 < q_0 \le \infty$. *Let* Φ *be as in (26).*
(i) *There exists a constant c such that*

$$\| (\langle f,\, \psi^e_{j,k}\rangle)_{j,k,e} \,|\ell_{p_0,\max(p_0,q_0)}\| \le c \, \| \, f \, |\dot{F}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)\| \tag{23}$$

holds for all $f \in \dot{F}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)$. *The estimate becomes false if one replaces* $\ell_{p_0,\max(p_0,q_0)}$ *by* $\ell_{p_0,u}$ *with* $0 < u < \max(p_0, q_0)$.
(ii) *Let* $p_0 < q_0$. *Then there exists a distribution* $f \in \dot{B}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)$ *such that the sequence* $(\langle f,\, \psi^e_{j,k}\rangle)_{j,k,e}$ *of its wavelet coefficients does not belong to* $\ell_{p_0,\infty}$.
(iii) *Let* $q_0 < p_0$. *For any* $u < p_0$ *there exists a distribution* $f \in \dot{B}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)$ *such that the sequence* $(\langle f,\, \psi^e_{j,k}\rangle)_{j,k,e}$ *of its wavelet coefficients does not belong to* $\ell_{p_0,u}$.

Remark 11. (i) Theorem 8 remains true if one replaces the homogeneous spaces either by the corresponding inhomogeneous spaces on $\mathbb{R}^d$ or the corresponding spaces on Ω. Of course, in such a case one has to work with different types of wavelet bases, see (32) and (39).
(ii) The statements in Theorem 8 are obvious in case $p_0 = q_0$. Then we simply have

$$\| (\langle f,\, \psi^e_{j,k}\rangle)_{j,k,e} \,|\ell_{p_0}\| \asymp \| \, f \, |\dot{F}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,p_0}(\mathbb{R}^d)\| \, ,$$

see Proposition 1. The interesting cases are those with $p_0 \neq q_0$. Again the spaces $\dot{F}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,\infty}(\mathbb{R}^d)$ have an extremal property. From all spaces $\dot{Y}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)$ with fixed p_0 they are the largest for which the estimate

$$\| (\langle f,\, \psi^e_{j,k}\rangle)_{j,k,e} \,|\ell_{p_0,\infty}\| \le c \, \| \, f \, |\dot{F}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)\| \tag{24}$$

holds for all $f \in \dot{Y}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}(\mathbb{R}^d)$, $Y \in \{F, B\}$.

4 Proofs

In this section all proofs will be given. However, also some additional material is presented. E.g., in Subsections 4.4, 4.7 properties and embeddings into approximation spaces will be discussed.

4.1 Wavelet isomorphisms

We recall a collection of results on the characterization of homogeneous and inhomogeneous distribution spaces. For the basics in wavelet theory we refer to [46].

4.1.1 Wavelets and homogeneous spaces

Let $\mathcal{P}$ denote the set of all polynomials on $\mathbb{R}^d$. Homogeneous spaces of Besov and Lizorkin-Triebel type are defined as subsets of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ modulo polynomials. This means we have to work with equivalence classes

$$[f] := \{g \in \mathcal{S}'(\mathbb{R}^d) : \, g = f + p \, , \, p \in \mathcal{P}\} \, , \qquad f \in \mathcal{S}'(\mathbb{R}^d). \tag{25}$$

We refer to [34, 18, 19, 41] for details. By a slight abuse of notation we will not distinguish between f and $[f]$ in general. The only exception will be Subsection 2.2. Usually the point of departure is a definition in Fourier analytic terms which we will not repeat. Here we are interested in a discretization by means of a wavelet transform.

Let ϕ denote an univariate scaling function associated with the wavelet ψ. We put $\psi^0 := \phi$. Let E denote the set of nonzero vertices of the unit cube in $\mathbb{R}^d$. For each vertex $e = (e_1, \ldots , e_d) \in E$ we let

$$\psi^e(x) := \psi^{e_1}(x_1) \, \ldots \, \psi^{e_d}(x_d) \, , \qquad x = (x_1, \ldots , x_d) \in \mathbb{R}^d \, .$$

We suppose that the collection Φ given by

$$\psi^e_{j,k}(x) := 2^{jd/2} \, \psi^e(2^j x - k) \, , \qquad e \in E \, , \quad k \in \mathbb{Z}^d, \quad j \in \mathbb{Z} \, , \tag{26}$$

forms an orthonormal basis for the space $L_2(\mathbb{R}^d)$. Furthermore we assume $\Phi \subset C^r(\mathbb{R}^d)$,

$$\sup_{x\in\mathbb{R}^d} (1 + |x|)^M \, |D^\alpha \psi^e(x)| < \infty \, , \qquad e \in E \, , \quad |\alpha| \le r \, , \tag{27}$$

and

$$\int_{\mathbb{R}^d} x^\alpha \, \psi^e(x) \, dx = 0 \, , \qquad e \in E \, , \quad |\alpha| < r \, , \tag{28}$$

for some $M > d$ and some $r > 0$. Let $\mathcal{X}$ be the characteristic function of the cube $[0, 1]^d$. Then we put

$$\mathcal{X}_{j,k}(x) := \mathcal{X}(2^j x - k) \, , \qquad x \in \mathbb{R}^d \, , \quad j \in \mathbb{N}_0 \, , k \in \mathbb{Z}^d \, .$$

In other words, $\mathcal{X}_{j,k}$ is the characteristic function of $Q_{j,k} := 2^{-j}\big([0, 1]^d + k\big)$. For the following two propositions we refer to [18], [19] and [30].

Proposition 1. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Further we suppose with $J := d/ \min(1, p, q)$*

$$r > \max\Big\{J - d - s, s\Big\} \qquad \textit{and} \qquad M > \max\Big\{J, d + r\Big\} .$$

Then, for every $f \in \dot{F}^s_{p,q}(\mathbb{R}^d)$*, we have*

$$f = \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E} \langle f,\, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\,, \tag{29}$$

convergence in $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ *(and in* $\dot{F}^s_{p,q}(\mathbb{R}^d)$ *if* $q<\infty$*), and*

$$\|\, f\,|\dot{F}^s_{p,q}(\mathbb{R}^d)\| \asymp \Big\| \Big(\sum_{j=-\infty}^{\infty} 2^{j(s+d/2)q} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E} |\langle f,\, \psi^e_{j,k}\rangle|^q\, \mathcal{X}_{j,k}(\,\cdot\,)\Big)^{1/q} \Big| L_p(\mathbb{R}^d)\Big\|\,. \tag{30}$$

Proposition 2. *Let* $s\in\mathbb{R}$*,* $0<p\le\infty$ *and* $0<q\le\infty$*. Further we suppose with* $J := d/\min(1,p)$

$$r > \max\Big\{J-d-s, s\Big\} \qquad and \qquad M > \max\Big\{J, d+r\Big\}\,.$$

Then, for every $f \in \dot{B}^s_{p,q}(\mathbb{R}^d)$*, we have*

$$f = \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E} \langle f,\, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\,,$$

convergence in $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ *(and in* $\dot{B}^s_{p,q}(\mathbb{R}^d)$ *if* $\max(p,q)<\infty$*), and*

$$\|\, f\,|\dot{B}^s_{p,q}(\mathbb{R}^d)\| \asymp \Big(\sum_{j=-\infty}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p})q}\Big(\sum_{k\in\mathbb{Z}^d} \sum_{e\in E} |\langle f,\, \psi^e_{j,k}\rangle|^p\Big)^{q/p}\Big)^{1/q}. \tag{31}$$

Remark 12. These two propositions require an interpretation. If ψ is the Meyer wavelet on $\mathbb{R}$, then this function satisfies a moment condition of infinite order, i.e., (28) holds for all α (and (27) for all $M>d$). This means that the coefficients $\langle f,\, \psi^e_{j,k}\rangle$ in (29) do not depend on the special representative f of the equivalence class $[f]$. In case that ψ satisfies a moment condition of finite order r only, then we have to calculate modulo polynomials of order less than r. Hence, we deal with a different set and also with a different topology. However, with the restrictions on s as given in the above propositions there are isomorphisms mapping the homogeneous spaces onto the sets of distributions modulo polynomials of order less than r and with (30) and (31) finite, respectively. We omit details.

4.1.2 Wavelets and inhomogeneous spaces

We use the same notation as in the previous subsection. In addition, we put

$$\psi^e_{-1,k}(x) := \phi(x_1-k_1)\,\ldots\,\phi(x_d-k_d)\,, \qquad x\in\mathbb{R}^d\,,\ k\in\mathbb{Z}^d\,,\ e\in E_{-1} := \{(0,\ldots,0)\}\,.$$

Furthermore we put $E_j := E$ for $j \geq 0$. In connection with inhomogeneous spaces on $\mathbb{R}^d$ our basic set Φ is now defined to be the collection of all functions

$$\psi^e_{j,k}\, , \qquad j \geq -1\, ,\ k \in \mathbb{Z}^d\, ,\ e \in E_j\, . \tag{32}$$

Then we have the following, see e.g. [43, Thm. 1.20].

Proposition 3. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Further we suppose with $J := d/\min(1,p,q)$*

$$r > \max\Big\{J-d-s, s\Big\} \qquad \textit{and} \qquad M > \max\Big\{J, d+r\Big\}.$$

Then, for every $f \in F^s_{p,q}(\mathbb{R}^d)$, we have

$$f = \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E_j} \langle f,\, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\, , \tag{33}$$

convergence in $\mathcal{S}'(\mathbb{R}^d)$ (and in $F^s_{p,q}(\mathbb{R}^d)$ if $q<\infty$), and

$$\|\, f\,|F^s_{p,q}(\mathbb{R}^d)\| \asymp \Big\| \Big(\sum_{j=-1}^{\infty} 2^{j(s+d/2)q} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E_j} |\langle f,\, \psi^e_{j,k}\rangle|^q\, \mathcal{X}_{j,k}(\,\cdot\,)\Big)^{1/q} \Big| L_p(\mathbb{R}^d)\Big\|. \tag{34}$$

Proposition 4. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Further we suppose with $J := d/\min(1,p)$*

$$r > \max\Big\{J-d-s, s\Big\} \qquad \textit{and} \qquad M > \max\Big\{J, d+r\Big\}.$$

Then, for every $f \in B^s_{p,q}(\mathbb{R}^d)$, we have

$$f = \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E_j} \langle f,\, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\, ,$$

convergence in $\mathcal{S}'(\mathbb{R}^d)$ (and in $B^s_{p,q}(\mathbb{R}^d)$ if $\max(p,q)<\infty$), and

$$\|\, f\,|B^s_{p,q}(\mathbb{R}^d)\| \asymp \Big(\sum_{j=-1}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p})q} \Big(\sum_{k\in\mathbb{Z}^d} \sum_{e\in E_j} |\langle f,\, \psi^e_{j,k}\rangle|^p \Big)^{q/p} \Big)^{1/q}. \tag{35}$$

4.1.3 Wavelets and domains

Let $\Omega \subset \mathbb{R}^d$ be a bounded and open set. Then we define the spaces $F^s_{p,q}(\Omega)$ and $B^s_{p,q}(\Omega)$ by restrictions, see e.g. [43, 2.1.1]. More exactly, we put

$$\begin{aligned} X^s_{p,q}(\Omega) &:= \Big\{ f \in D'(\Omega): \quad f = g_{|\Omega} \quad \text{for some } g \in X^s_{p,q}(\mathbb{R}^d)\Big\} \\ \|\, f\,|X^s_{p,q}(\Omega)\| &:= \inf \|\, g\,|X^s_{p,q}(\mathbb{R}^d)\|\, , \end{aligned}$$

where the infimum is taken over all $g \in X^s_{p,q}(\mathbb{R}^d)$ such that $f = g_{|\Omega}$. Here $X \in \{F, B\}$.

For our purpose it is enough to observe the following. Let the univariate scaling function ϕ and the associated wavelet ψ be compactly supported, say

$$\Big(\operatorname{supp}\phi \,\cup\, \operatorname{supp}\psi\Big) \subset [-N, N]$$

for some $N > 0$. For given $f \in X^s_{p,q}(\Omega)$ let $\mathcal{E}f$ denote an extension of f such that

$$\|\, \mathcal{E}f \,|X^s_{p,q}(\mathbb{R}^d)\| \le 2\, \|\, f \,|X^s_{p,q}(\Omega)\| \le 2\, \|\, \mathcal{E}f \,|X^s_{p,q}(\mathbb{R}^d)\| \, .$$

Then

$$\mathcal{E}f = \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{e\in E_j} \langle \mathcal{E}f,\, \psi^e_{j,k}\rangle \, \psi^e_{j,k} \, .$$

Hence, also

$$\mathcal{E}^* f := \sum_{j=-1}^{\infty} \sum_{\operatorname{supp}\psi^e_{j,k}\cap\Omega\neq\emptyset} \sum_{e\in E_j} \langle \mathcal{E}f,\, \psi^e_{j,k}\rangle \, \psi^e_{j,k} \tag{36}$$

is an extension of f such that

$$\|\, \mathcal{E}^* f \,|X^s_{p,q}(\mathbb{R}^d)\| \asymp \|\, f \,|X^s_{p,q}(\Omega)\| \, . \tag{37}$$

Moreover, we have

$$\operatorname{supp}\mathcal{E}^* f \subset \Gamma := \{x \in \mathbb{R}^d :\ \operatorname{dist}(x, \Omega) < 2N\} \, . \tag{38}$$

For inhomogeneous spaces on bounded domains we define Φ to be the collection of all functions $\psi^e_{j,k}$ such that

$$\Omega \cap \operatorname{supp}\psi^e_{j,k} \neq \emptyset \, , \qquad j \ge -1,\ k \in \mathbb{Z}^d \, , \ e \in E_j \, . \tag{39}$$

With (36) and (37) we do not get an intrinsic characterization of $X^s_{p,q}(\Omega)$. Those characterizations are only known under more specific restrictions on the domain Ω and on the set of admissible parameters, see e.g. the monographs [6] and [43].

4.2 Sequence spaces

The described characterizations of the Lizorkin-Triebel and Besov spaces allow a discretization of our problem, i.e., it is enough to investigate best m-term approximation on appropriate sequence spaces. By ignoring the finite sum $\sum_{e\in E_j}$ we are lead to the following type of sequence spaces.

Definition 1. *Let $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $\nabla = (\nabla_j)_j$ be a sequence of nontrivial subsets of $\mathbb{Z}^d$.*

(i) *Let $0 < p \le \infty$. Then $b^s_{p,q}(\nabla)$ consists of all sequences $a = (a_{j,\lambda})_{j,\lambda}$ such that*

$$\| \, a \, | b^s_{p,q}(\nabla) \| := \Big(\sum_{j=0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \Big(\sum_{\lambda \in \nabla_j} |a_{j,\lambda}|^p \Big)^{q/p} \Big)^{1/q} < \infty \, . \tag{40}$$

(ii) *Let $0 < p < \infty$. Then $f^s_{p,q}(\nabla)$ consists of all sequences $a = (a_{j,\lambda})_{j,\lambda}$ such that*

$$\| \, a \, | f^s_{p,q}(\nabla) \| := \Big\| \Big(\sum_{j=0}^{\infty} \sum_{\lambda \in \nabla_j} 2^{j(s+d/2)q} \, |a_{j,\lambda}|^q \, \mathcal{X}_{j,\lambda}(\,\cdot\,) \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| < \infty \, . \tag{41}$$

(iii) *We define $f^s_{\infty,\infty}(\nabla) := b^s_{\infty,\infty}(\nabla)$.*

Remark 13. (i) Two special cases of sequences ∇ are of particular importance. The first one is simply $\nabla_j = \mathbb{Z}^d$ for all j, and we will denote the corresponding spaces by $b^s_{p,q}$ and $f^s_{p,q}$, respectively. For the second one, let $\Omega \subset \mathbb{R}^d$ be a bounded open (nontrivial) set. Then, by looking at (39), we define ∇ by

$$\nabla_j := \Big\{ k \in \mathbb{Z}^d \, : \, \psi^e_{j,k} \in \Phi \text{ for some } e \in E_j \Big\} \, . \tag{42}$$

The corresponding sequence spaces will be denoted by $b^s_{p,q}(\Omega)$ and $f^s_{p,q}(\Omega)$, respectively. Furthermore, the following fact is immediate. There exist positive constants C_1 and C_2 and an appropriate nonnegative integer J, such that

$$C_1 \le 2^{-jd} \, |\nabla_j| \le C_2 \, , \qquad j \ge J. \tag{43}$$

(ii) Let $\nabla_j := \mathbb{Z}^d$ for all j. If the summation in (40) ((41)) extends over $\mathbb{Z}$ with respect to j, then we will call the spaces homogeneous and denote them by $\dot{b}^s_{p,q}$ ($\dot{f}^s_{p,q}$).
(iii) Obviously $b^s_{p,p}(\nabla) = f^s_{p,p}(\nabla)$ and $\dot{b}^s_{p,p} = \dot{f}^s_{p,p}$, respectively.
(iv) The straightforward extension of part (ii) of the above definition to $p = \infty$ does not lead to the correct spaces in general, see e.g. Frazier and Jawerth [18]. The only exception is the case $q = \infty$.

Later on we shall need the following duality assertion.

Lemma 4. *Let $1 < p < \infty$, $1 < q < \infty$, and $s \in \mathbb{R}$. Let ∇ be as in Remark 13 (i). Then*

$$(\dot{f}^s_{p,q})' = \dot{f}^{-s}_{p',q'} \qquad \textit{and} \qquad (f^s_{p,q}(\nabla))' = f^{-s}_{p',q'}(\nabla) \, ,$$

where q' is defined by $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof. The counterparts of these duality relations for function spaces can be found in Triebel [41, 2.11.2] (inhomogeneous spaces) and in Frazier and Jawerth [18] (homogeneous spaces). This has to be combined with the wavelet isomorphisms described in the previous subsection. ■

4.3 Gagliardo-Nirenberg inequalities and embeddings of sequence spaces

In the context of Gagliardo-Nirenberg type inequalities Ohru, see Brezis and Mironescu [3], has proved the following nice inequality for sequences.
Let $0 < \Theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, and $0 < q \leq \infty$. We put $s := (1-\Theta)s_0 + \Theta s_1$. Let I either stand for $\mathbb{N}_0$ or for $\mathbb{Z}$. Then there exists a constant c such that

$$\Big(\sum_{j\in I} 2^{jsq}\,|a_j|^q\Big)^{1/q} \leq c\,\Big(\sup_{j\in I} 2^{js_0}\,|a_j|\Big)^{1-\Theta}\,\Big(\sup_{j\in I} 2^{js_1}\,|a_j|\Big)^{\Theta} \tag{44}$$

holds for all sequences $(a_j)_j$ of complex numbers. Although the proof given there is only stated for $I = \mathbb{N}_0$, it can be carried over to the case $I = \mathbb{Z}$ by obvious modifications. For given $0 < p_0, p_1 \leq \infty$ we define p by $1/p := (1-\Theta)/p_0 + \Theta/p_1$. Further, putting

$$a_j := \sum_{\lambda\in\nabla_j} |a_{j,\lambda}|\,\mathcal{X}_{j,\lambda}(x)$$

for x fixed, applying (44) and Hölder's inequality we arrive at the following Gagliardo-Nirenberg type inequalities.

Lemma 5. *Let $0 < \Theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, and $0 < q \leq \infty$. We put $s := (1-\Theta)s_0 + \Theta s_1$. For given $0 < p_0, p_1 \leq \infty$ we define p by $1/p := (1-\Theta)/p_0 + \Theta/p_1$.*
(i) *Let $p_1 < \infty$. There exists a constant c such that*

$$\|\, a\,|\dot{f}^{s}_{p,q}\| \leq c\,\|\, a\,|\dot{f}^{s_0}_{p_0,\infty}\|^{1-\Theta}\,\|\, a\,|\dot{f}^{s_1}_{p_1,\infty}\|^{\Theta} \tag{45}$$

holds for all $a \in \dot{f}^{s_0}_{p_0,\infty} \cap \dot{f}^{s_1}_{p_1,\infty}$.
(ii) *Let $p_1 < \infty$. There exists a constant c such that*

$$\|\, a\,|f^{s}_{p,q}(\nabla)\| \leq c\,\|\, a\,|f^{s_0}_{p_0,\infty}(\nabla)\|^{1-\Theta}\,\|\, a\,|f^{s_1}_{p_1,\infty}(\nabla)\|^{\Theta} \tag{46}$$

holds for all $a \in f^{s_0}_{p_0,\infty}(\nabla) \cap f^{s_1}_{p_1,\infty}(\nabla)$.

On the basis of these Gagliardo-Nirenberg inequalities one could easily prove the counterparts of the embeddings stated in Lemma 1 on the sequence space level.

However, by means of Propositions 1-4 they are obvious. Now we turn to the proof of some embeddings for real interpolation spaces. For the basics in real interpolation we refer to [1, 2, 40].

Theorem 9. *Let* $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $0 < \Theta < 1$, *and*

$$s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}\,. \tag{47}$$

We put $s = (1-\Theta)s_0 + \Theta s_1$ *and* $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$. *Then*

$$\dot{f}^s_{p,\infty} \hookrightarrow \Big(\dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\Big)_{\Theta,\infty} \qquad \textit{and} \qquad f^s_{p,\infty}(\nabla) \hookrightarrow \Big(f^{s_0}_{p_0,q_0}(\nabla), f^{s_1}_{p_1,q_1}(\nabla)\Big)_{\Theta,\infty}\,. \tag{48}$$

Proof. We only deal with the homogeneous spaces. The modifications needed for the classes $f^s_{p,q}(\nabla)$ are obvious.

Step 1. Our main tool will be the following well-known assertion in interpolation theory, see [1, Prop. 5.2.10].

Lemma 6. *Let* $\{X_0, X_1\}$ *be an interpolation couple of Banach spaces and let* X *be an intermediate space. Furthermore, let* $0 < \Theta < 1$. *Then the embedding* $(X_0, X_1)_{\Theta,1} \hookrightarrow X$ *holds if, and only if, for some positive constant* c *the estimate*

$$\|\, f\,|X\| \le c\,\|\, f\,|X_0\|^{1-\Theta}\,\|\, f\,|X_1\|^{\Theta}$$

is fulfilled for all $f \in X_0 \cap X_1$.

We need a few preparations.

Step 2. Let $\varepsilon > 0$. For a given sequence $a := (a_{j,\lambda})$ we define the sequences $|a|$ and $|a|^\varepsilon$ by

$$|a| := \big(|a_{j,\lambda}|\big)_{j,\lambda} \qquad \text{and} \qquad |a|^\varepsilon := \big(|a_{j,k}|^\varepsilon\big)_{j,\lambda}\,,$$

respectively. Obviously

$$\|\,|a|\,|\dot{f}^s_{p,q}\| = \|\,a\,|\dot{f}^s_{p,q}\| \qquad \text{and} \qquad \|\,|a|^\varepsilon\,|\dot{f}^{s\varepsilon}_{p/\varepsilon,q/\varepsilon}\| = \|\,|a|\,|\dot{f}^s_{p,q}\|^\varepsilon\,. \tag{49}$$

We claim

$$K\big(t, a, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big) = K\big(t, |a|, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)\,, \qquad t > 0\,, \tag{50}$$

where $K(t, a, X_0, X_1)$ denotes the K-functional with respect to the pair (X_0, X_1). We argue as follows. Let $|a| = a_0 + a_1$, $a_i = (a_{i,j,\lambda})_{j,\lambda} \in \dot{f}^{s_i}_{p_i,q_i}$, $i = 0,1$. Then, for appropriate $\varphi_{j,\lambda}$, we find

$$a_{j,\lambda} = |a_{j,\lambda}|\, e^{i\varphi_{j,\lambda}} = (a_{0,j,\lambda} + a_{1,j,\lambda})\, e^{i\varphi_{j,\lambda}} = \widetilde{a}_{0,j,\lambda} + \widetilde{a}_{1,j,\lambda}\,,$$

where $\widetilde{a}_{i,j,\lambda} := a_{i,j,\lambda}\, e^{i\varphi_{j,\lambda}}$, $i = 0, 1$. Because of

$$\| \, (\widetilde{a}_{i,j,\lambda})_{j,\lambda} \, | \dot{f}^{s_i}_{p_i,q_i} \| = \| \, (a_{i,j,\lambda})_{j,\lambda} \, | \dot{f}^{s_i}_{p_i,q_i} \| \, , \qquad i = 0, 1,$$

we obtain

$$K\big(t, a, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big) \le K\big(t, |a|, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big) \, .$$

For the converse estimate we use a similar argument based on $|a_{j,\lambda}| = a_{j,\lambda}\, e^{-i\varphi_{j,\lambda}}$ with the same numbers $\varphi_{j,\lambda}$. We get

$$K\big(t, |a|, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big) \le K\big(t, a, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big) \, .$$

This proves the claim. As an immediate conclusion we find

$$\| \, |a| \, | \big(\dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)_{\theta,q} \| = \| \, a \, | \big(\dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)_{\theta,q} \| \, . \tag{51}$$

Step 3. We need a further preparation. Let $0 < \varepsilon < 1$. We claim

$$K\big(t, |a|^{\varepsilon}, \dot{f}^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}, \dot{f}^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big) \asymp K\big(t^{1/\varepsilon}, |a|, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)^{\varepsilon} \tag{52}$$

where the constants "behind" $\asymp$ do not depend on a. The proof of this estimate is based on the lattice structure of the spaces $\dot{f}^{s}_{p,q}$, i.e.

$$a, b \in \dot{f}^{s}_{p,q}, \ |a| \le |b| \qquad \text{implies} \qquad \big\|a\big|\dot{f}^{s}_{p,q}\big\| \le \big\|b\big|\dot{f}^{s}_{p,q}\big\| \, . \tag{53}$$

This property allows us to impose certain restrictions on the sequences a_0 and a_1 used for the representation $|a| = a_0 + a_1$ in the definition of $K\big(t^{1/\varepsilon}, |a|, \dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)$. E.g., we may restrict ourselves to sequences of nonnegative real numbers. This may be seen as follows. As $|a|$ itself is a sequence of real numbers, from $|a| = a_0 + a_1$ it follows immediately that $|a| = \Re a_0 + \Re a_1$. Since the estimates $|\Re a_0| \le |a_0|$, $|\Re a_1| \le |a_1|$ are always true we may assume a_0 and a_1 to be real. Furthermore, if there are pairs (j, m), such that $a_{0,j,m} < 0$ then we define

$$\widetilde{a}_{0,j',m'} = \begin{cases} a_{0,j',m'}, & (j',m') \ne (j,m), \\ 0, & (j',m') = (j,m), \end{cases} \qquad \widetilde{a}_{1,j',m'} = \begin{cases} a_{1,j',m'}, & (j',m') \ne (j,m), \\ |a_{j,m}|, & (j',m') = (j,m) \, . \end{cases}$$

Clearly we have $|\widetilde{a}_0| \le |a_0|$, $|\widetilde{a}_1| \le |a_1|$ and $|a| = \widetilde{a}_0 + \widetilde{a}_1$. This procedure is iterated until $\widetilde{a}_0$ and $\widetilde{a}_1$ are nonnegative. The same argument can be applied to $|a|^{\varepsilon}$ and $K\big(t, |a|^{\varepsilon}, \dot{f}^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}, \dot{f}^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big)$. By means of these considerations we find

$$\begin{aligned}
K\big(t^{1/\varepsilon}, |a|, f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)^{\varepsilon} &= \inf_{|a|=a_0+a_1} \Big(\big\|a_0\big|f^{s_0}_{p_0,q_0}\big\| + t^{1/\varepsilon}\big\|a_1\big|f^{s_1}_{p_1,q_1}\big\|\Big)^{\varepsilon} \\
&\asymp \inf_{|a|=a_0+a_1} \Big(\big\||a_0|^{\varepsilon}\big|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\||a_1|^{\varepsilon}\big|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big) \\
&= \inf_{|a|=a_0^{1/\varepsilon}+a_1^{1/\varepsilon}} \Big(\big\|a_0\big|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\|a_1\big|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big) \\
&\asymp \inf_{|a|^{\varepsilon}=a_0+a_1} \Big(\big\|a_0\big|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\|a_1\big|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big) = K\big(t, |a|^{\varepsilon}, f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}, f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big).
\end{aligned}$$

The second equivalence is a consequence of

$$\begin{aligned}
\inf_{|a|=a_0^{1/\varepsilon}+a_1^{1/\varepsilon}} & \Big(\big\|a_0|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\|a_1|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big) \\
& \geq \inf_{(1/2)(a_0+a_1)\leq|a|^\varepsilon\leq a_0+a_1} \Big(\big\|a_0|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\|a_1|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big) \\
& = \inf_{|a|^\varepsilon=a_0+a_1} \Big(\big\|a_0|f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}\big\| + t\big\|a_1|f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big\|\Big)\,,
\end{aligned}$$

where the last equation is due to the lattice property (53). Similarly one obtains the reverse estimate. Altogether this proves (52).

Step 4. Now we are in position to apply Lemma 6. Temporarily we assume $1 \leq p_0 < p_1 < \infty$. Observe, that $p_0 < p_1$ together with (47) imply

$$\dot{f}^{s_0}_{p_0,\infty} \cap \dot{f}^{s_1}_{p_1,\infty} = \dot{f}^{s_0}_{p_0,\infty} \hookrightarrow \dot{f}^{s}_{p,1} \hookrightarrow \dot{f}^{s_1}_{p_1,\infty} = \dot{f}^{s_0}_{p_0,\infty} + \dot{f}^{s_1}_{p_1,\infty}$$

(in the sense of equivalent norms), see Lemma 1(i) and Proposition 1. As an immediate conclusion of (46) and Lemma 6 we obtain

$$(\dot{f}^{s_0}_{p_0,\infty}, \dot{f}^{s_1}_{p_1,\infty})_{\Theta,1} \hookrightarrow \dot{f}^{s}_{p,1}\,. \tag{54}$$

Step 5. We assume $1 < p_0, p_1, q_0, q_1 < \infty$. We continue by applying duality arguments. To this end it will be convenient for us to replace (54) by its weaker version

$$(\dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1})_{\Theta,1} \hookrightarrow \dot{f}^{s}_{p,1}\,.$$

Since

$$\big(X_0, X_1\big)'_{\Theta,1} = \big(X_0', X_1'\big)_{\Theta,\infty}$$

(here (X_0, X_1) has to be an interpolation pair of Banach spaces such that $X_0 \cap X_1$ is dense in X_0 as well as in X_1, see [40, Thm. 1.11.2]), we conclude from Lemma 4

$$\dot{f}^{-s}_{p',\infty} \hookrightarrow (\dot{f}^{-s_0}_{p_0',q_0'}, \dot{f}^{-s_1}_{p_1',q_1'})_{\Theta,\infty}\,. \tag{55}$$

Concerning the required density we only mention that finite sequences are dense in $\dot{f}^{s}_{p,q}$ if $\max(p, q) < \infty$. Since $(X_0, X_1)_{\Theta,\infty} = (X_1, X_0)_{1-\Theta,\infty}$ by a change of notation we obtain (48) under the restrictions $1 < p_0 < p_1 < \infty$, $1 < q_0, q_1 < \infty$ and s_0, s_1 as in (47).

Step 6. We remove the restrictions with respect to p_0, p_1, q_0 and q_1. Fix $0 < \varepsilon < \min(1, p_0, p_1, q_0, q_1)$. Then we derive from Step 3

$$\begin{aligned}
\big\||a|^\varepsilon\big|\big(f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}, f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big)_{\Theta,\infty}\big\| & \asymp \sup_{t>0} t^{-\Theta} K\big(t^{1/\varepsilon}, |a|, f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)^\varepsilon \\
= \Big(\sup_{s>0} s^{-\Theta} K\big(s, |a|, f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)\Big)^\varepsilon & = \big\||a|\big|\big(f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)_{\Theta,\infty}\big\|^\varepsilon
\end{aligned}$$

and hence with Step 5 and (51)

$$\big\|a\big|f^s_{p,\infty}\big\|^\varepsilon = \big\||a|^\varepsilon\big|f^{s\varepsilon}_{p/\varepsilon,q/\varepsilon}\big\| \gtrsim \big\||a|^\varepsilon\big|\big(f^{s_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}, f^{s_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}\big)_{\theta,\infty}\big\|$$
$$\asymp \big\||a|\big|\big(f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)_{\theta,\infty}\big\|^\varepsilon = \big\|a\big|\big(f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big)_{\theta,\infty}\big\|^\varepsilon .$$

This completes the proof. ■

4.4 Approximation spaces associated to sequence spaces

Approximation spaces with respect to best m-term approximation have been defined in Subsection 2.3. Later on, we shall need the fact that approximation spaces have nice properties with respect to real interpolation. The following is proved in [36], see also [4].

Proposition 5. *Let $0 < u, u_0, u_1 \le \infty$ and $0 < \Theta < 1$. Further we assume $s_0, s_1 > 0$ and $s_0 \ne s_1$. Let X be a quasi-Banach space and $\mathcal{D}$ a subset of X. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$, it holds*

$$(\mathcal{A}^{s_0}_{u_0}(X,\mathcal{D}), \mathcal{A}^{s_1}_{u_1}(X,\mathcal{D}))_{\Theta,u} = \mathcal{A}^s_u(X,\mathcal{D}) . \tag{56}$$

Most important for us will be the study of certain sequence spaces. In this connection we concentrate on best m-term approximation with respect to the canonical orthonormal basis of $\ell_2(I)$, where I is a fixed infinite index set. We put

$$\mathcal{B} := \{e^j : \quad j \in I\}, \quad e^j := (e^j_k)_k , \quad e^j_k := \delta_{j,k} , \; j,k \in I.$$

Remark 14. When dealing with approximation in the sequence spaces from Definition 1, the observation

$$\sigma_m\big(a, x^s_{p,q}, \mathcal{B}\big) = \inf\Big\{\big\|\, a - \sum_{j\in\Lambda} a_j\, e^j \,\big|\, x^s_{p,q}\big\| : \quad |\Lambda| \le m\Big\}, \qquad x \in \{b,f\}, \tag{57}$$

where $a = (a_j)_{j\in I}$ and $I = \mathbb{N}_0 \times \mathbb{Z}^d$, is most helpful. With other words, the optimal approximation is always given by a partial sum of $a = \sum_{j\in I} a_j\, e^j$. This follows immediately from the lattice property of the b- and f-spaces.

By $\ell_{p,u}(I)$ we denote the Lorentz sequence spaces. They are the collection of all sequences $a = (a_j)_{j\in I}$, such that

$$\big\|a\big|\ell_{p,u}(I)\big\| := \Big\|\big(n^{\frac{1}{p}-\frac{1}{u}} a^*_n\big)_{n\in\mathbb{N}}\Big|\ell_u(\mathbb{N})\Big\| < \infty, \qquad 0 < p, u \le \infty ,$$

where $a^* = (a^*_n)_n$ denotes the non-increasing rearrangement of a. Our point of departure is the following nice result of Pietsch [37, Ex. 1].

Proposition 6. *Let $0 < p_1, u \le \infty$. Let I be a fixed index set. Then $a \in \ell_{p_1}(I)$ belongs to the approximation space $\mathcal{A}^s_u(\ell_{p_1}(I), \mathcal{B})$ if and only if $a \in \ell_{p_0,u}(I)$ where $1/p_0 := s + 1/p_1$. Furthermore,*

$$\| \, a \, | \mathcal{A}^s_u(\ell_{p_1}(I), \mathcal{B}) \| \asymp \| \, a \, | \ell_{p_0,u}(I) \| \, . \tag{58}$$

4.5 Approximation spaces and embeddings into approximation spaces

To begin with, we formulate a discrete counterpart of (3) which is in our context a consequence of Proposition 6.

Corollary 3. *Let ∇ be as in Definition 1. Let $0 < p_0 < p_1 \le \infty$. Then*

$$\mathcal{A}^{\frac{1}{p_0} - \frac{1}{p_1}}_{p_0}(\ell_{p_1}(\mathbb{Z} \times \mathbb{Z}^d), \mathcal{B}) = \dot{b}^{d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0}$$

as well as

$$\mathcal{A}^{\frac{1}{p_0} - \frac{1}{p_1}}_{p_0}(\ell_{p_1}(\mathbb{N}_0 \times \nabla), \mathcal{B}) = b^{d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0}(\nabla) \, ,$$

both in the sense of equivalent quasi-norms.

Proof. Observe $\dot{b}^{d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0} = \ell_{p_0,p_0} = \ell_{p_0}$. ■

Another variant is given by the following.

Corollary 4. *Let ∇ be as in Definition 1. Let $0 < p_0 < p_1 \le \infty$ and $s \in \mathbb{R}$. Then*

$$\mathcal{A}^{\frac{1}{p_0} - \frac{1}{p_1}}_{p_0}(\dot{b}^{s+d(\frac{1}{p_1} - \frac{1}{2})}_{p_1,p_1}, \mathcal{B}) = \dot{b}^{s+d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0}$$

as well as

$$\mathcal{A}^{\frac{1}{p_0} - \frac{1}{p_1}}_{p_0}(b^{s+d(\frac{1}{p_1} - \frac{1}{2})}_{p_1,p_1}(\nabla), \mathcal{B}) = b^{s+d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0}(\nabla) \, ,$$

both in the sense of equivalent quasi-norms.

Proof. We consider the mapping $a \mapsto b$ defined by $b_{j,\lambda} = 2^{js} \, a_{j,\lambda}$. Then

$$\sigma_m(a, \mathcal{B})_{\dot{b}^{s+d(\frac{1}{p_1} - \frac{1}{2})}_{p_1,p_1}} = \sigma_m(b, \mathcal{B})_{\dot{b}^{d(\frac{1}{p_1} - \frac{1}{2})}_{p_1,p_1}} = \sigma_m(b, \mathcal{B})_{\ell_{p_1}}$$

and

$$\| \, a \, | \dot{b}^{s+d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0} \| = \| \, b \, | \dot{b}^{d(\frac{1}{p_0} - \frac{1}{2})}_{p_0,p_0} \| = \| \, b \, | \ell_{p_0} \| \, .$$

This proves the claim. ■

Remark 15. A case of particular interest is given by $s + d(\frac{1}{p_1} - \frac{1}{2}) = 0$. Then we find

$$\mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{b}^0_{p_1,p_1}, \mathcal{B}) = \dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0}$$

as well as

$$\mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(b^0_{p_1,p_1}(\nabla), \mathcal{B}) = b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0}(\nabla)\,,$$

again in the sense of equivalent quasi-norms.

Within a scale $\mathcal{A}^s_q(X, \mathcal{D})$ most interesting are the spaces with $q = \infty$.

Lemma 7. *Let* $0 < p_0 < p_1 \le \infty$. *We have*

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty} \hookrightarrow \mathcal{A}_{\infty}^{\frac{1}{p_0}-\frac{1}{p_1}}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big)\,, \tag{59}$$

as well as

$$f^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty}(\nabla) \hookrightarrow \mathcal{A}_{\infty}^{\frac{1}{p_0}-\frac{1}{p_1}}\big(b^0_{p_1,p_1}(\nabla), \mathcal{B}\big)\,. \tag{60}$$

Proof. Let $0 < u_0 < p_0 < u_1 < p_1$. From Remark 15 we conclude

$$\dot{f}^{d(1/u_0-1/p_1)}_{u_0,u_0} \hookrightarrow \mathcal{A}_{\infty}^{1/u_0-1/p_1}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big) \qquad \text{and} \qquad \dot{f}^{d(1/u_1-1/p_1)}_{u_1,u_1} \hookrightarrow \mathcal{A}_{\infty}^{1/u_1-1/p_1}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big)\,.$$

Let $\frac{1}{p_0} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$. Then Theorem 9 combined with Proposition 5 implies

$$\begin{aligned}\dot{f}^{d(1/p_0-1/p_1)}_{p_0,\infty} &\hookrightarrow \Big(\dot{f}^{d(1/u_0-1/p_1)}_{u_0,u_0}, \dot{f}^{d(1/u_1-1/p_1)}_{u_1,u_1}\Big)_{\theta,\infty} \\ &\hookrightarrow \Big(\mathcal{A}_{\infty}^{1/u_0-1/p_1}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big), \mathcal{A}_{\infty}^{1/u_1-1/p_1}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big)\Big)_{\theta,\infty} = \mathcal{A}_{\infty}^{1/p_0-1/p_1}\big(\dot{b}^0_{p_1,p_1}, \mathcal{B}\big).\end{aligned}$$

This proves (59). The proof of (60) uses the same type of arguments. ■

Kyriazis [29] has proved that the approximation spaces associated to Lizorkin-Triebel spaces do not depend on the microscopic parameter q (in the sense of equivalent quasi-norms). Hence, as an immediate conclusion of Lemma 7 we obtain the following generalization.

Theorem 10. *Let* $0 < p_0 < p_1 \le \infty$ *and* $0 < q_1 \le \infty$. *We have*

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty} \hookrightarrow \mathcal{A}_{\infty}^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{f}^0_{p_1,q_1}, \mathcal{B})\,, \tag{61}$$

as well as

$$f^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty}(\nabla) \hookrightarrow \mathcal{A}_{\infty}^{\frac{1}{p_0}-\frac{1}{p_1}}(f^0_{p_1,q_1}(\nabla), \mathcal{B})\,. \tag{62}$$

Proof. Kyriazis [29] has dealt with the homogeneous spaces only. However, more or less obvious modifications yield the same result for the spaces $f^s_{p,q}(\nabla)$. ■

4.6 The asymptotic behaviour of the widths of best m-term approximation

Now we turn to the behaviour of the quantities $\sigma_m(Y, X, \mathcal{B})$.

4.6.1 Estimates from below

Let ∇ be as in Remark 13(i). For the estimates from below we shall discuss two types of sequences.

Example 1. Let $a^m := (a^m_{j,\lambda})_{j,\lambda}$, $m \in \mathbb{N}$, and

$$a^m_{j,\lambda} := \begin{cases} 2^{-jd(\frac{1}{2}-\frac{1}{p_1})} & j = j_m\,, \quad \lambda \in \Lambda_m\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{63}$$

where j_m is chosen such that $|\nabla_{j_m}| \geq 2m$ and Λ_m is a subset of ∇_{j_m} satisfying $|\Lambda_m| = 2m$. An easy calculation shows

$$\big\| \, a^m \, | b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \big\| = \big\| \, a^m \, \big| f^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \big\| = (2m)^{1/p_0}\,.$$

Due to the special structure of the sequences, the best m-term approximation is easy to determine. With either $X = f^0_{p_1,q_1}(\nabla)$ or $X = b^0_{p_1,q_1}(\nabla)$ we obtain

$$\sigma_m(a^m, \mathcal{B})_X = m^{1/p_1}\,.$$

Hence

$$\sigma_m\Big(y^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla), x^0_{p_1,q_1}(\nabla), \mathcal{B}\Big) \geq c\, m^{-1/p_0+1/p_1}\,, \tag{64}$$

where $x, y \in \{f, b\}$ and c is independent of m.

Example 2. Let $b^m := (b^m_{j,\lambda})_{j,\lambda}$, $m \in \mathbb{N}$. This time the construction is a little bit more sophisticated. Let m be fixed. We choose a sequence of pairwise disjoint cubes Q_{j,k^j}, $j = 1, \ldots, 2m$, and define

$$b^m_{j,\lambda} = \begin{cases} 2^{-jd(\frac{1}{2}-\frac{1}{p_1})} & 1 \leq j \leq m, \quad \lambda = k^j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{65}$$

Similarly b^{2m} is defined (taking the same sequence of cubes). As a consequence of this construction we get

$$\big\| \, b^m \, | b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \big\| = m^{1/q_0} \qquad \text{and} \qquad \big\| \, b^m \, \big| f^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \big\| = m^{1/p_0}$$

as well as

$$\big\| b^{2m} - b^m | f^0_{p_1,q_1}(\nabla) \big\| = m^{1/p_1} \qquad \text{and} \qquad \big\| b^{2m} - b^m | b^0_{p_1,q_1}(\nabla) \big\| = m^{1/q_1}\,.$$

Furthermore, b^m is a best m-term approximation for b^{2m}. This implies the estimates

$$\sigma_m\Big(b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla), f^0_{p_1,q_1}(\nabla), \mathcal{B}\Big) \geq m^{-1/q_0+1/p_1}\,, \tag{66}$$

$$\sigma_m\Big(f^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla), b^0_{p_1,q_1}(\nabla), \mathcal{B}\Big) \geq m^{-1/p_0+1/q_1}\,, \tag{67}$$

$$\sigma_m\Big(b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla), b^0_{p_1,q_1}(\nabla), \mathcal{B}\Big) \geq m^{-1/q_0+1/q_1}\,. \tag{68}$$

This proves the needed estimates from below for the widths σ_m related to pairs of spaces $(y^{s_0}_{p_0,q_0}(\Omega), x^{s_1}_{p_1,q_1}(\Omega))$ with $x, y \in \{b, f\}$, at least if Ω contains a cube with side length 2. A simple modification of the b^m (one takes a sequence of $2m$ pairwise disjoint cubes Q_{j,k^j}, $j = j_0, \ldots, j_0+2m$, but starting at the higher level $j_0 = j_0(\Omega)$) yield the estimates (66) - (68) (up to a positive constant $c = c(\Omega)$) also for smaller Ω. Practically the same examples can be used for the pairs $(y^{s_0}_{p_0,q_0}, x^{s_1}_{p_1,q_1})$ and $(\dot{y}^{s_0}_{p_0,q_0}, \dot{x}^{s_1}_{p_1,q_1})$ with $x, y \in \{b, f\}$. This carries over to the distribution spaces by means of Propositions 1 - 4 and the comments given in Subsection 2.5.

4.6.2 Proofs of Theorems 1 - 4 and Lemma 2

Proof of Theorem 1 . The claim is an immediate consequence of Theorem 10 (estimate from above) and the inequality (64) (estimate from below). ■

Proof of Theorem 2 . *Step 1.* Suppose $p_0 < q_0 < p_1$. Under the given restrictions on q_0 the Jawerth-Franke embedding, see Lemma 1 (iii), yields

$$\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \dot{f}^{d(\frac{1}{q_0}-\frac{1}{p_1})}_{q_0,q_1} \hookrightarrow \dot{f}^{d(\frac{1}{q_0}-\frac{1}{p_1})}_{q_0,\infty}\,.$$

Theorem 1 combined with obvious monotonicity properties of the σ_m lead to

$$\sigma_m\Big(\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}, \dot{f}^0_{p_1,q_1}, \mathcal{B}\Big) \lesssim \sigma_m\Big(\dot{f}^{d(\frac{1}{q_0}-\frac{1}{p_1})}_{q_0,\infty}, \dot{f}^0_{p_1,q_1}, \mathcal{B}\Big) \lesssim m^{-(\frac{1}{q_0}-\frac{1}{p_1})}\,.$$

For the estimate from below we use (66).

Step 2. Let $q_0 = p_1$. Then we know $\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_1} \hookrightarrow \dot{f}^0_{p_1,q_1}$ which implies

$$\sigma_m\Big(\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_1}, \dot{f}^0_{p_1,q_1}, \mathcal{B}\Big) \lesssim \sigma_m\Big(\dot{f}^0_{p_1,q_1}, \dot{f}^0_{p_1,q_1}, \mathcal{B}\Big) \lesssim 1.$$

The estimates from below are also given in (66).

Step 3. Suppose $0 < q_0 < p_0$. The estimate from above is a consequence of

$$\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty}$$

and Theorem 1. Furthermore, (64) yields the estimate from below. ■

Proof of Theorem 3. *Step 1.* Let $p_1 \le q_1$. The continuous embedding

$$\dot{f}^0_{p_1,p_1} \hookrightarrow \dot{b}^0_{p_1,q_1}\,,$$

the monotonicity of σ_m and Theorem 1 yield the estimate from above. The estimate from below is covered by (64).

Step 2. Let $p_0 \le q_1 < p_1$. A lifting argument as used in Corollary 4 in combination with Theorem 10 yield

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}_\infty^{\frac{1}{p_0}-\frac{1}{q_1}}\big(\dot{f}^{d(\frac{1}{q_1}-\frac{1}{p_1})}_{q_1,q_1},\mathcal{B}\big)\,.$$

Now we use the continuous embedding $\dot{f}^{d(\frac{1}{q_1}-\frac{1}{p_1})}_{q_1,q_1} \hookrightarrow \dot{b}^0_{p_1,q_1}$ to conclude

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}_\infty^{\frac{1}{p_0}-\frac{1}{q_1}}\big(\dot{b}^0_{p_1,q_1},\mathcal{B}\big)\,.$$

This proves the estimate from above. The estimate from below follows from (67). ■

Proof of Theorem 4. *Step 1.* The estimate from above in the limiting case, i.e. the case

$$\frac{1}{p_0}-\frac{1}{p_1}=\frac{1}{q_0}-\frac{1}{q_1}\,,$$

has been proved by Kyriazis [29, Thm. 3.6]. His arguments, originally given in the homogeneous situation, carry over to the inhomogeneous case as well. The step where he refers to some previously proven Jackson-type inequality can be replaced by the usage of Proposition 6 (recall $b^{d(\frac{1}{p}-\frac{1}{2})}_{p,p} = \ell_p(\mathbb{N}_0\times\mathbb{Z}^d)$).

Step 2. Let

$$\frac{1}{p_0}-\frac{1}{p_1}<\frac{1}{q_0}-\frac{1}{q_1}\,,\qquad \text{i.e.}\qquad r=\frac{1}{p_0}-\frac{1}{p_1}\,.$$

We define q_* by $\frac{1}{q_*} := \frac{1}{p_0}-\frac{1}{p_1}+\frac{1}{q_1} < \frac{1}{q_0}$. Hence $q_0 < q_*$. This implies

$$\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_*} \hookrightarrow \mathcal{A}_\infty^{\frac{1}{p_0}-\frac{1}{p_1}}\big(\dot{b}^0_{p_1,q_1},\mathcal{B}\big)\,,$$

where the second embedding follows from Step 1.

Step 3. It remains to investigate the case

$$\frac{1}{q_0}-\frac{1}{q_1}<\frac{1}{p_0}-\frac{1}{p_1}\,,\qquad \text{i.e.}\qquad r=\frac{1}{q_0}-\frac{1}{q_1}\,.$$

We define p_* by $\frac{1}{p_*} := \frac{1}{q_0} - \frac{1}{q_1} + \frac{1}{p_1} < \frac{1}{p_0}$. Hence $p_0 < p_*$. By Lemma 1(ii) and Step 1 we conclude

$$\dot{b}_{p_0,q_0}^{d(\frac{1}{p_0}-\frac{1}{p_1})} \hookrightarrow \dot{b}_{p_*,q_0}^{d(\frac{1}{p_*}-\frac{1}{p_1})} \hookrightarrow \mathcal{A}_\infty^{\frac{1}{q_0}-\frac{1}{q_1}}\big(\dot{b}^0_{p_1,q_1},\mathcal{B}\big)\,.$$

Step 4. All estimates from below follow from (64) and (68). ■

Proof of Lemma 2. The lemma is an immediate consequence of Lemma 1 and (64). ■

4.6.3 Proof of Corollaries 1, 2

Proof of Corollary 1. By using the interpretation in front of the Corollary it becomes just a reformulation of Theorems 1, 2 with $q_1 = 2$ and $s_1 = 0$. ■

Proof of Corollary 2. *Step 1.* Estimate from above. In the sense of our interpretation we have the continuous embedding $\dot{F}^0_{1,1}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$, more precisely, the inequality

$$\|\, T[f]\,|L_1(\mathbb{R}^d)\| \lesssim \|\,[f]\,|\dot{F}^0_{1,1}(\mathbb{R}^d)\| \tag{69}$$

holds for all $[f] \in \dot{F}^0_{1,1}(\mathbb{R}^d)$. By using the Fourier analytic definition of $\dot{F}^0_{1,1}(\mathbb{R}^d)$, see e.g. [25], this is obvious. The above inequality (69) guarantees

$$\sigma_m\Big(\dot{Y}_{p_0,q_0}^{d(\frac{1}{p_0}-1)}(\mathbb{R}^d), L_1(\mathbb{R}^d), \Phi\Big) \lesssim \sigma_m\Big(\dot{Y}_{p_0,q_0}^{d(\frac{1}{p_0}-1)}(\mathbb{R}^d), \dot{F}^0_{1,1}(\mathbb{R}^d), \Phi\Big)$$

with $Y \in \{F, B\}$. Hence, Theorems 1, 2 yield the upper bounds in (14) and (15).
Step 2. Estimate from below. This time we use

$$\|\,[f]\,|\dot{B}^0_{1,\infty}(\mathbb{R}^d)\| \lesssim \|\, T[f]\,|L_1(\mathbb{R}^d)\|\,.$$

Then

$$\sigma_m\Big(\dot{Y}_{p_0,q_0}^{d(\frac{1}{p_0}-1)}(\mathbb{R}^d), \dot{B}^0_{1,\infty}(\mathbb{R}^d), \Phi\Big) \lesssim \sigma_m\Big(\dot{Y}_{p_0,q_0}^{d(\frac{1}{p_0}-1)}(\mathbb{R}^d), L_1(\mathbb{R}^d), \Phi\Big)$$

follows. Now we apply Theorem 3 with $t = 1$ and Theorem 4 with

$$r = \min\Big(\frac{1}{p_0} - 1, \frac{1}{q_0}\Big) = \frac{1}{p_0} - 1\,.$$

This completes the proof. ■

4.6.4 Proof of Theorem 5

By using our interpretation from Subsection 2.2 and Corollary 1 we obtain $\dot{F}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_\infty(L_{p_1}(\mathbb{R}^d),\Phi)$ as well as $\dot{B}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_\infty(L_{p_1}(\mathbb{R}^d),\Phi)$ with $q_0 \le p_0 < p_1$. But then

$$\dot{B}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{B}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0}(\mathbb{R}^d) = \dot{F}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,\infty}(\mathbb{R}^d)$$

follows. ■

4.6.5 Proof of Theorem 6

Step 1. Estimate from above in (19). Similarly as in the proof of Corollary 2, we have the continuous embedding $B^0_{1,1}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$, and hence the estimate

$$\sigma_m\Big(B^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_1(\mathbb{R}^d),\Phi\Big) \lesssim \sigma_m\Big(B^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), \dot{B}^0_{1,1}(\mathbb{R}^d),\Phi\Big)\,.$$

Now Theorem 4 yields the upper bounds in (19).

Step 2. The estimates from below are similar to the ones in Examples 1 and 2. For simplicity, we only treat the case $d=1$, the generalization to $d>1$ is immediate.

Substep 2.1. Let the scaling function ϕ and the associated wavelet ψ be compactly supported with $(\operatorname{supp}\phi \cup \operatorname{supp}\psi) \subset [-N,N]$. Then we define functions f_m by

$$f_m := \sum_{l=1}^{m} \phi(\cdot - 4lN)\,.$$

Obviously, the summands have pairwise disjoint support, and for any $\psi^e_{j,k} \in \Phi$ there is at most one $l \in \{1,\dots,m\}$, such that $\operatorname{supp}\psi^e_{j,k} \cap \operatorname{supp}\phi(\cdot - 4lN) \neq \emptyset$. At first, we find

$$\big\|f_m\big|F^{\frac{1}{p_0}-1}_{p_0,q_0}(\mathbb{R})\big\| \asymp \big\|f_m\big|B^{\frac{1}{p_0}-1}_{p_0,q_0}(\mathbb{R})\big\| \asymp m^{1/p_0}\,.$$

Furthermore, f_m is a best m-term approximation of f_{2m}, and hence

$$\sigma_m\Big(Y^{\frac{1}{p_0}-1}_{p_0,q_0}(\mathbb{R}), L_1(\mathbb{R}),\Phi\Big) \gtrsim m^{-\frac{1}{p_0}}\big\|f_{2m}-f_m\big|L_1(\mathbb{R})\big\| = m^{-\frac{1}{p_0}+1}\big\|\phi\big|L_1(\mathbb{R})\big\|\,.$$

Substep 2.2. The estimate

$$\sigma_m\Big(B^{d(\frac{1}{p_0}-1)}_{p_0,q_0}(\mathbb{R}^d), L_1(\mathbb{R}^d),\Phi\Big) \ge c\, m^{-1/q_0+1}$$

with some positive c follows by an appropriate modification of Example 2.

Step 3. Estimate from above in (20).

Let

$$\varepsilon := \frac{1}{2}\Big(s_0 - s_1 - d\Big(\frac{1}{p_0}-\frac{1}{p_1}\Big)\Big)\,.$$

Then, with $x, y \in \{b, f\}$, we find

$$y^{s_0}_{p_0,q_0} \hookrightarrow b^{s_0-\varepsilon}_{p_0,p_0} = \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0}\Big(b^{s_0-\varepsilon-d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_1,p_1}, \mathcal{B}\Big) \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_{\infty}(x^{s_1}_{p_1,q_1}, \mathcal{B}),$$

where we have used Corollary 4 and elementary monotonicity properties of the approximation spaces. By means of Proposition 4 this can be transfered to the associated distribution spaces.

Step 4. Estimate from below. We use Example 1 with $j_m = 0$ for all m. ■

4.6.6 Proof of Theorem 7

Recall, the sequence spaces $f^s_{p,q}(\Omega)$ and $b^s_{p,q}(\Omega)$ have been defined in Remark 13. Our proof relies on the following result taken from [9].

Proposition 7. *Let* $0 < p_0, p_1, q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}^d$ *s.t.*

$$s_0 - s_1 > d \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big).$$

holds. Let $\Omega \subset \mathbb{R}^d$ *be a bounded, open and nontrivial. Then we have*

$$\sup\Big\{\sigma_n(a, \mathcal{B})_{b^{s_1}_{p_1,q_1}(\Omega)} : \quad \| a\|_{b^{s_0}_{p_0,q_0}(\Omega)} \le 1\Big\} \asymp n^{-\frac{s_0-s_1}{d}}. \tag{70}$$

Remark 16. We have referred above to [9] but we wish to mention that the proof given there made essential use of some ideas taken from [12]. Moreover, there is a series of forerunners on the level of functions spaces, we refer to the surveys [10, 44] and [31].

Step 1. Preparations. Thanks to the elementary embeddings

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d),$$

see, e.g. [41, Prop. 2.3.2/2], we may concentrate on $X = Y = B$.

Step 2. Estimate from above. Recall, W sends f to the sequence $(\langle f, \psi^e_{j,k}\rangle)_{j,k,e}$ of its wavelet coefficients. Recall also, that $\mathcal{E}^* f$ denotes an extension of f satisfying (36)-(38). Hence, with $f \in B^{s_0}_{p_0,q_0}(\Omega)$, we find $Wf \in b^{s_0}_{p_0,q_0}(\Gamma)$ (for Γ see (38)). Furthermore,

$$\begin{aligned} \| f - \sum_{(j,k,e)\in\Lambda_m} \langle \mathcal{E}^* f, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\, |B^{s_1}_{p_1,q_1}(\Omega)\| \;&\lesssim\; \| \mathcal{E}^* f - \sum_{(j,k,e)\in\Lambda_m} \langle \mathcal{E}^* f, \psi^e_{j,k}\rangle\, \psi^e_{j,k}\, |B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| \\ &\lesssim\; \| (\langle \mathcal{E}^* f, \psi^e_{j,k}\rangle)_{(j,k,e)\notin\Lambda_m}\, |b^{s_1}_{p_1,q_1}(\Gamma)\|. \end{aligned}$$

But this implies

$$\sigma_m(f,\Phi)_{B^{s_1}_{p_1,q_1}(\Omega)} \lesssim \sigma_m(W\mathcal{E}^*f,\mathcal{B})_{b^{s_1}_{p_1,q_1}(\Gamma)}\,,$$

which yields the estimate from above in view of Proposition 7.

Step 3. Estimate from below. We can argue as in Step 2 using this time Proposition 7 with respect to a set Γ' such that $\Gamma' \subset \Omega$ and $\mathrm{dist}\,(\Gamma',\partial\Omega) > 0$. One may also work with the test sequences from [9] directly. We omit details. ■

4.7 More on approximation spaces

Also in this subsection, we assume that ∇ is as in Remark 13.

Again interpolation theory will be our main tool. To begin with, we need a little supplement of a famous theorem of Lions and Peetre on the interpolation of vector-valued L_p-spaces, see [33]. With $L_p(A)$ we denote the Lebesgue-space of functions with values in a Banach-space A.

Lemma 8. *Let $1 < p_0, p_1 < \infty$, $0 < \Theta < 1$ and $1 < p \le q < \infty$, where*

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,.$$

Further, let $\{A_0, A_1\}$ be an interpolation couple of Banach-spaces. Then

$$L_p\big((A_0,A_1)_{\Theta,q}\big) \hookrightarrow \big(L_{p_0}(A_0), L_{p_1}(A_1)\big)_{\Theta,q}\,.$$

Proof. The classical result of Lions and Peetre is just

$$L_p\big((A_0,A_1)_{\Theta,p}\big) = \big(L_{p_0}(A_0), L_{p_1}(A_1)\big)_{\Theta,p} \tag{71}$$

in the sense of equivalent norms. For us, it is convenient to refer to the proof given in [40, Thm. 1.18.4]. Step 1 and Step 2 of this proof contain the proof of (71). One has to modify the estimates in formula (6) only. After having started with $\|\, v(x)\, |\big(L_{p_0}(A_0), L_{p_1}(A_1)\big)_{\Theta,q}\|^p$, one has to continue with an application of the Minkowski inequality with respect to $\|\,\cdot\,|L_{q/p}(0,\infty)\|$. ■

Now we turn to a supplement of Theorem 9.

Theorem 11. *Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}^d$, $s_0 \neq s_1$, $0 < \Theta < 1$, and $0 < p \le q < \infty$, where $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$. We put $s = (1-\Theta)s_0 + \Theta s_1$. Then*

$$\dot{f}^s_{p,q} \hookrightarrow \big(\dot{f}^{s_0}_{p_0,q_0}, \dot{f}^{s_1}_{p_1,q_1}\big)_{\Theta,q} \qquad \textit{and} \qquad f^s_{p,q}(\nabla) \hookrightarrow \Big(f^{s_0}_{p_0,q_0}(\nabla), f^{s_1}_{p_1,q_1}(\nabla)\Big)_{\Theta,q}$$

Proof. Again we concentrate on the proof of the homogeneous case.

Step 1. This time our proof relies on an interpolation formula for sequence spaces. We define $\dot{\ell}^s_q$ to be the collection of all sequences $a = (a_j)_j$ of complex numbers such that

$$\| a \,|\dot{\ell}^s_q\| := \| (2^{js} \,|a_j|)_j \,|\ell_q(\mathbb{Z})\| < \infty \,.$$

Then the following interpolation formula

$$\dot{\ell}^s_q = (\dot{\ell}^{s_0}_{q_0}, \, \dot{\ell}^{s_1}_{q_1})_{\Theta,q} \tag{72}$$

holds, where $0 < q, q_0, q_1 \le \infty$, $s = (1-\Theta)\, s_0 + \Theta\, s_1$, and $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$. A proof can be found in [2, Thm. 5.6.1], for the inhomogeneous case we also refer to [40, Thm. 1.18.2].

Step 2. Let $1 < p_0, p_1, q_0, q_1, q < \infty$. Just by definition $f^s_{p,q}$ is isomorphic to a closed subspace of $L_p(\ell^s_q)$. Now we employ the method of retraction and coretraction, see [2, Thm. 6.4.2] or [40, Thm. 1.2.4]. The claim becomes a consequence of (72) in combination with Lemma 8.

Step 3. To remove the restriction on the parameters p_0, p_1, q_0, q_1, q we use the same arguments as in Theorem 9. ■

Now this information can be used to derive additional knowledge about the embeddings of $\dot{f}^s_{p,q}$ into approximation spaces, similarly to Lemma 7.

Theorem 12. *Let* $0 < p_0 < p_1 < \infty$ *and* $0 < q_0, q_1, q \le \infty$. *Then*

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_q(\dot{f}^0_{p_1,q_1}, \mathcal{B}) \qquad or \qquad \dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_q(\dot{f}^0_{p_1,q_1}(\nabla), \mathcal{B}) \tag{73}$$

holds if, and only if $\max(p_0, q_0) \le q$.

Proof. Again we concentrate on the proof of the homogeneous case.

Step 1. Sufficiency in (73). *Substep 1.1.* Let $q_0 \le p_0$. From Corollary 4 and the monotonicity of the approximation spaces, see Remark 6, we derive

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0} = \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0}(\dot{f}^0_{p_1,q_1}, \mathcal{B}) \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_q(\dot{f}^0_{p_1,q_1}, \mathcal{B}) \,.$$

Substep 1.2. Let $p_0 < q_0$. Let $0 < u_0 < p_0 < u_1 < p_1 < \infty$. Applying Corollary 4 and Theorem 10 we find

$$\big(\dot{f}^{d(\frac{1}{u_0}-\frac{1}{p_1})}_{u_0,u_0}, \dot{f}^{d(\frac{1}{u_1}-\frac{1}{p_1})}_{u_1,\infty}\big)_{\Theta,q_0} \hookrightarrow \big(\mathcal{A}^{\frac{1}{u_0}-\frac{1}{p_1}}_{u_0}(\dot{f}^0_{p_1,q_1}, \mathcal{B}), \mathcal{A}^{\frac{1}{u_1}-\frac{1}{p_1}}_{\infty}(\dot{f}^0_{p_1,q_1}, \mathcal{B})\big)_{\Theta,q_0} \tag{74}$$

for any $0 < \Theta < 1$. Now we choose Θ such that $\frac{1}{p_0} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$. Proposition 5 implies that the space on the right-hand side of (74) is $\mathcal{A}_{q_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{f}^0_{p_1,q_1}, \mathcal{B})$. Since

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \big(\dot{f}^{d(\frac{1}{u_0}-\frac{1}{p_1})}_{u_0,u_0}, \dot{f}^{d(\frac{1}{u_1}-\frac{1}{p_1})}_{u_1,\infty}\big)_{\Theta,q_0},$$

see Theorem 12, we have proved

$$\dot{f}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}_{q_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{f}^0_{p_1,q_1}, \mathcal{B}).$$

Again the monotonicity of the spaces $\mathcal{A}^s_q(X,\mathcal{D})$ with respect to q completes the proof.

Step 2. Necessity. *Substep 2.1.* A preparation. Again we employ the isomorphism $a \mapsto b$ defined by $b_{j,\lambda} := 2^{jd(\frac{1}{2}-\frac{1}{p_1})}\, a_{j,\lambda}$, see Corollary 4. Then $\|\, a\,|\dot{f}^0_{p_1,p_1}\| = \|\, b\,|\ell_{p_1}\|$. This combined with the independence of $\mathcal{A}^s_q(\dot{f}^0_{p_1,q_1}, \mathcal{B})$ from q_1, see [29], yields

$$\dot{y}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}_{q}^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{f}^0_{p_1,q_1}, \mathcal{B}) \tag{75}$$

if, and only if

$$\dot{y}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0} \hookrightarrow \mathcal{A}_{q}^{\frac{1}{p_0}-\frac{1}{p_1}}(\ell_{p_1}, \mathcal{B})$$

with $y \in \{f,b\}$. Taking into account Proposition 6, then we see, that (75) is equivalent to

$$\dot{y}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0} \hookrightarrow \ell_{p_0,q}, \qquad y \in \{f,b\}. \tag{76}$$

Substep 2.2. We define a sequence $a := (a_{j,k})_{j,k}$ by

$$a_{j,k} := \begin{cases} 2^{-j\frac{d}{p_0}}\, c_j & j \in \mathbb{N},\ 0 \le k_i < 2^j,\ i = 1,\ldots d; \\ 0 & \text{otherwise}. \end{cases}$$

Here $c := (c_j)_j$ denotes a sequence of real numbers which will be chosen later on. Our definition yields

$$\|\, a\,|\dot{f}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}\| = \|\, c\,|\ell_{q_0}\|. \tag{77}$$

Let $M_0 := 0$ and $M_j := \sum_{u=1}^{j} 2^{jd}$, $j \in \mathbb{N}$. Next we suppose that the sequence $(2^{-j\frac{d}{p_0}}\, c_j)_{j\in\mathbb{N}}$ is nonincreasing. By a^*_n we denote the elements of the nonincreasing rearrangement of the sequence a. Now we are able to calculate $\|\, a\,|\ell_{p_0,q}\|$. Indeed, we obtain

$$\begin{aligned} \|\, a\,|\ell_{p_0,q}\| &= \Big(\sum_{n=1}^{\infty}\Big(n^{\frac{1}{p_0}-\frac{1}{q}}\, a^*_n\Big)^q\Big)^{1/q} \\ &\asymp \sum_{j=1}^{\infty}\sum_{n=M_{j-1}+1}^{M_j}\Big(M_j^{\frac{1}{p_0}-\frac{1}{q}}\, 2^{-j\frac{d}{p_0}}\, c_j\Big)^q\Big)^{1/q} \\ &\asymp \|\, c\,|\ell_q\|. \end{aligned} \tag{78}$$

A comparison of (76), (77), and (78) yields $q_0 \le q$.

Substep 2.3. Let $(Q_{j,k^j})_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint dyadic cubes. We define a sequence $a := (a_{j,k})_{j,k}$ by

$$a_{j,k} := \begin{cases} c_j & j \in \mathbb{N}\,,\ k = k^j\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where the sequence $c := (c_j)_j$ will be chosen later. Then

$$\| \, a \, | \dot{f}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0} \| = \| \, c \, | \ell_{p_0} \| \, .$$

For $c_j \ge 0$, nonincreasing, we obtain

$$\| \, a \, | \ell_{p_0,q} \| = \Big(\sum_{n=1}^{\infty} \Big(n^{\frac{1}{p_0}-\frac{1}{q}} \, c_n \Big)^q \Big)^{1/q} = \| \, c \, | \ell_{p_0,q} \| \, .$$

Since $\ell_{p_0} \hookrightarrow \ell_{p_0,q}$ if, and only if $p_0 \le q$ we have proved the necessity of $q \ge \max(p_0, q_0)$. ■

Finally we also want to treat embeddings of the spaces $\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}$ and $b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla)$ into corresponding approximation spaces.

Theorem 13. *Let $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1, q \le \infty$. Then*

$$\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}_q^{\frac{1}{p_0}-\frac{1}{p_1}}(\dot{f}^0_{p_1,q_1}, \mathcal{B}) \qquad \text{or} \qquad b^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0}(\nabla) \hookrightarrow \mathcal{A}_q^{\frac{1}{p_0}-\frac{1}{p_1}}(f^0_{p_1,q_1}(\nabla), \mathcal{B})$$

holds if, and only if, $q_0 \le p_0 \le q$.

Proof. *Step 1.* Let $q_0 \le p_0 \le q$. Then we obtain

$$\dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \dot{b}^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,p_0} = \mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(f^0_{p_1,q_1}, \mathcal{B}) \hookrightarrow \mathcal{A}_q^{\frac{1}{p_0}-\frac{1}{p_1}}(f^0_{p_1,q_1}, \mathcal{B})\,,$$

due to Corollary 4 (Remark 15) and the monotonicity of the ℓ_q-spaces.

Step 2. It remains to deal with the necessity of the restrictions $q_0 \le p_0 \le q$. It will be convenient for us to use Substep 2.1 of the proof of Theorem 12.

Substep 2.1. Concerning the necessity of $p_0 \le q$, it is enough to notice that $\| \, a \, | \dot{b}^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0} \| = \| \, c \, | \ell_{p_0} \|$ and $\| \, a \, | \ell_{p_0,q} \| = \| \, c \, | \ell_{p_0,q} \|$, whenever

$$a_{j,k} := \begin{cases} c_k & j = 0\,,\ k \in \mathbb{Z}^d\,; \\ 0 & \text{otherwise} \end{cases}$$

for some sequence $c = (c_k)_k$. As in Substep 2.3 of the proof of Theorem 12, the embedding $\ell_{p_0} \hookrightarrow \ell_{p_0,q}$ is necessary and hence enforces $q \ge p_0$.

Substep 2.2. We assume $q_0 > p_0$. Let $(k_j)_{j\in\mathbb{N}} \subset \mathbb{Z}^d$ be some arbitrary fixed sequence. Then define a sequence $a := (a_{j,k})_{j,k}$ by

$$a_{j,k} := \begin{cases} c_j & j \in \mathbb{N}\,,\ k = k^j\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where the sequence $c := (c_j)_j$ consists of positive real numbers and is nonincreasing. Hence

$$\|\, a\,|b^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0}\| = \|\, c\,|\ell_{q_0}\| \qquad \text{and} \qquad \|\, a\,|\ell_{p_0,q}\| = \|\,(n^{1/p_0-1/q}c_n)_n\,|\ell_q\|\,.$$

Choosing $c_n := n^{-1/p_0}$, $n \in \mathbb{N}$, we have $c \in \ell_{q_0}$, but $c \notin \ell_{p_0,q}$ for any $q < \infty$. If $q = \infty$, then we modify the definition of c by taking this time $c_n := n^{-1/p_0}\,\log(1+n)$, $n \in \mathbb{N}$. ∎

Remark 17. While the picture concerning embeddings into approximation spaces $A^s_u(\dot f^0_{p,q}, \mathcal{B})$ and $A^s_u(\dot F^0_{p,q}(\mathbb{R}^d), \Phi)$ and their respective modifications is complete, little is known about approximation spaces with respect to Besov spaces, apart from the special case $p = q$ and Theorem 4. However, we want to mention at least one result of Jawerth and Milman [27]. They proved

$$A^r_{q_r}\big(\dot B^0_{p,q}(\mathbb{R}^d), \Phi\big) = \dot B^{dr}_{(p_r,q_r),q_r}(\mathbb{R}^d)$$

where $r > 0$, $0 < p, q \le \infty$, $\frac{1}{p_r} := r + \frac{1}{p}$, and $\frac{1}{q_r} := r + \frac{1}{q}$. The spaces $\dot B^s_{(p,q),r}(\mathbb{R}^d)$ are Besov-Lorentz spaces (Besov spaces defined on Lorentz spaces instead of Lebesgue spaces).

4.8 Proof of Theorem 8

Step 1. Proof of (i). Combining Propositions 1, 2 with the results obtained in Substep 2 of the proof of Theorem 12 we get the equivalence of

$$\dot y^{d(\frac{1}{p_0}-\frac{1}{2})}_{p_0,q_0} \hookrightarrow \ell_{p_0,q}\,, \qquad y \in \{f, b\}$$

and

$$\dot Y^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_q(\dot F^0_{p_1,q_1}, \Phi)\,, \qquad Y \in \{F, B\}.$$

Since

$$\dot F^{d(\frac{1}{p_0}-\frac{1}{p_1})}_{p_0,q_0} \hookrightarrow \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_q(\dot F^0_{p_1,q_1}, \Phi)$$

holds if, and only if $\max(p_0, q_0) \le q$, see Theorem 12, we have proved (23).

Step 2. To prove (ii) and (iii) we use Theorem 13 in combination with Proposition 2 and argue as in Step 1. ∎

References

[1] C. Bennett, R. Sharpley, *Interpolation of operators.* Academic Press, Boston, 1988.

[2] J. Bergh, J. Löfström, *Interpolation Spaces.* Springer, Berlin, 1976.

[3] H. Brezis, P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. *J. Evol. Eq.* **1** (2001), 387–404.

[4] P.L. Butzer, K. Scherer, *Approximationsprozesse und Interpolationsmethoden.* Mannheim, Zürich, 1968.

[5] M. Christ, The extension problem for certain function spaces involving fractional orders of differentiability. *Ark. Mat.* **22** (1984), 63-81.

[6] A. Cohen, *Numerical Analysis of Wavelet Methods.* Elsevier Science, Amsterdam, 2003.

[7] A. Cohen, R. DeVore, R. Hochmuth, Restricted nonlinear approximation. *Constr. Approx.* **16** (2000), 85-113.

[8] S. Dahlke, W. Dahmen, R. DeVore, *Nonlinear approximation and adaptive techniques for solving elliptic operator equations*, in Multilevel wavelet methods for partial differential equations (ed. by A. Kurdila, W. Dahmen and P. Oswald), pp. 237-283, Academic Press, San Diego, 1997.

[9] S. Dahlke, E. Novak, W. Sickel, Optimal approximation of elliptic problems by linear and nonlinear mappings II. *J. Complexity* **22**, 549–603.

[10] R.A. DeVore, Nonlinear Approximation. *Acta numerica* **7** (1998), 51-150.

[11] R.A. DeVore, B. Jawerth, and V. Popov, Compression of wavelet decompositions. *Amer. J. Math.* **114** (1992), 737–785.

[12] R.A. DeVore, G. Kyriazis, D. Leviatan and V.M. Tikhomirov, Wavelet compression and nonlinear n-widths. *Adv. Comput. Math.* **1** (1993), 197–214.

[13] R.A. DeVore, G. Petrova and V.N. Temlyakov, Best basis selection for approximation in L_p. *Found. Comp. Math.* **3** (2003), 161–185.

[14] R.A. DeVore, R.C. Sharpley, *Maximal functions measuring smoothness*. Mem. AMS 47 **293** (1984), 1–115..

[15] S.J. Dilworth, N.J. Kalton, D. Kutzarova, and V.N. Temlyakov, The thresholding greedy algorithm, greedy bases, and duality. *Constr. Approx.* **19** (2003), 575–597.

[16] D.E. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*. Cambridge Univ. Press, Cambridge, 1996.

[17] J. Franke, On the spaces $F^s_{p,q}$ of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains. *Math. Nachr.* **125** (1986), 29-68.

[18] M. Frazier, B. Jawerth, A discrete transform and decomposition of distribution spaces. *J. Functional Anal.* **93** (1990), 34-170.

[19] M. Frazier, B. Jawerth, G. Weiss, *Littlewood-Paley theory and the study of function spaces*. CBMS-AMS Regional Conf. Series **79**, 1991.

[20] R. Gribonval, M. Nielsen, Some remarks on non-linear approximation with Schauder bases. *East J. on Approx.* **7** (2001), 1–19.

[21] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, New Jersey, 2004.

[22] M. Hansen, W. Sickel, Best m-term approximation and function spaces of dominating mixed smoothness. Preprint, Jena, 2009.

[23] M. Hansen, J. Vybiral, The Jawerth-Franke embedding of spaces of dominating mixed smoothness. Georg. J. Math. (to appear).

[24] C.C. Hsiao, B. Jawerth, B.J. Lucier, X.M. Yu, Near optimal compression of orthonormal wavelet expansions, in *Wavelets: mathematics and applications*, pp. 425-446, CRC, Boca Raton, 1994.

[25] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces. *Math. Scand.* **40** (1977), 94-104.

[26] B. Jawerth, M. Milman, Weakly rearrangement invariant spaces and approximation by largest elements. Contemporary Math. **445** (2007), 103-110.

[27] B. Jawerth, M. Milman, Wavelets and best approximation in Besov spaces. *Isr. Math. Conf. Proc.* **5** (1992), 107–112.

[28] G. Kerkyacharian, D. Picard, Entropy, universal coding, approximation, and bases properties. *Constr. Approx.* **20** (2004), 1-37.

[29] G. Kyriazis, Non-linear approximation and interpolation spaces. *JAT* **113** (2001), 110-126.

[30] G. Kyriazis and P. Petrushev, New bases for Triebel-Lizorkin and Besov spaces. *Trans. AMS* **354** (2002), no. 2, 749-776.

[31] P. Oswald, On the degree of nonlinear spline approximation in Besov-Sobolev spaces. *JAT* **61** (1990), 131-157.

[32] P. Oswald, Greedy algorithms and best m-term approximation with respect to biorthogonal systems. *J. Fourier Anal. Appl.* 7(4) (2001), 325-341.

[33] J.L. Lions and J. Peetre, Sur une classe d'espaces d'interpolation, *Inst. Hautes Etudes Sci. Publ. Math.* **19** (1964), 5-68.

[34] J. Peetre, *New thoughts on Besov spaces.* Duke Univ. Math. Series I, Durham, N.C. (1976).

[35] J. Peetre, On spaces of Triebel-Lizorkin type. *Ark. Mat.* **13** (1975), 123–130.

[36] J. Peetre and G. Sparr, Interpolation of normed abelian groups. *Ann. Mat. Pura Appl.* **92** (1972), 217-262.

[37] A. Pietsch, Approximation spaces. *JAT* **32** (1980), 115-134.

[38] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type. *ZAA* **14** (1995), 105-140.

[39] H. Triebel *Spaces of Besov-Hardy-Sobolev type.* Teubner-Texte Math. 15, Teubner, Leipzig, 1978.

[40] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin 1978.

[41] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

[42] H. Triebel, *Theory of function spaces II*, Birkhäuser, Basel, 1992.

[43] H. Triebel, *Function spaces and wavelets on domains.* EMS Publishing House, Zürich, 2008.

[44] V.N. Temlyakov, Nonlinear methods of approximation. *Found. Comput. Math.* **3** (2003), 33-107.

[45] J. Vybiral, A new proof of the Jawerth-Franke Embedding. *Rev. Math. Comp.* **21** (2008), no. 1, 75–82.

[46] P. Wojtaszczyk, *A mathematical introduction to wavelets.* Cambridge Univ. Press, Cambridge, 1997.

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[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.