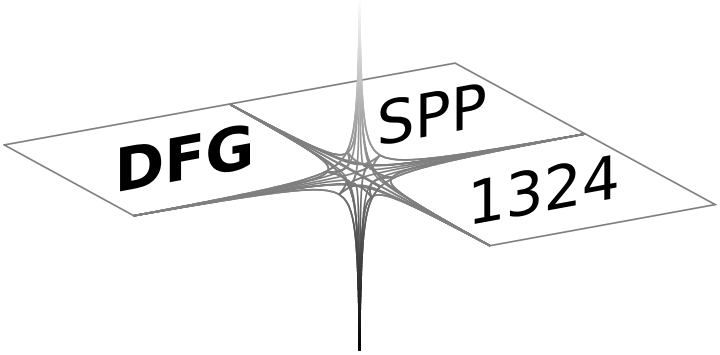
DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$

B. Niu, F.J. Hickernell, T. Müller-Gronbach, K. Ritter

Preprint 40



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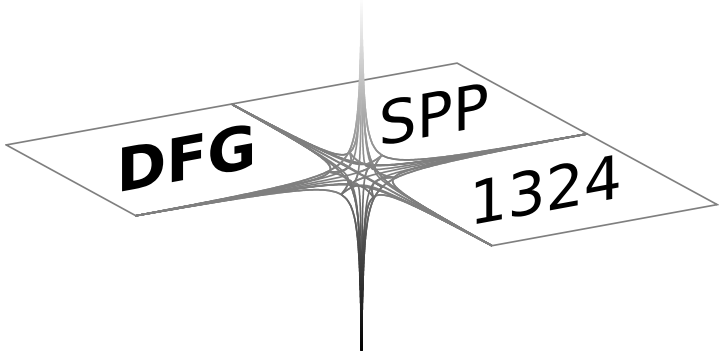

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Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$

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Abstract

Pricing a path-dependent financial derivative, such as an Asian option, requires the computation of $\mathbb{E}(g(B))$, the expectation of a payoff function g, that depends on a Brownian motion B. Employing a standard series expansion of B the latter problem is equivalent to the computation of the expectation of a function of the corresponding i.i.d. sequence of random coefficients. This motivates the construction and the analysis of algorithms for numerical integration with respect to a product probability measure on the sequence space $\mathbb{R}^{\mathbb{N}}$. The class of integrands studied in this paper is the unit ball in a reproducing kernel Hilbert space obtained by superposition of weighted tensor product spaces of functions of finitely many variables. Combining tractability results for high-dimensional integration with the multi-level technique we obtain new algorithms for infinite-dimensional integration. These deterministic multi-level algorithms use variable subspace sampling and they are superior to any deterministic algorithm based on fixed subspace sampling with respect to the respective worst case error.

Key words: infinite-dimensional quadrature, multi-level method, tractability, low discrepancy points, fixed subspace sampling, variable subspace sampling, minimal errors

1. Introduction

Infinite-dimensional quadrature problems, i.e., numerical integration with respect to measures on infinite-dimensional spaces, naturally arise for instance in the context of stochastic processes. A common approach to such

quadrature problems involves some kind of truncation or projection onto finite-dimensional subspaces, and accordingly the integrands have to be evaluated only at points from these subspaces.

This article is part of a recent line of research on infinite-dimensional quadrature problems, where the cost of function evaluation is assumed to be dimension dependent and where deterministic as well as randomized algorithms are studied. See [1] for integration on separable Banach spaces and [12, 5, 10] for integration on the sequence space $\mathbb{R}^{\mathbb{N}}$. In the latter papers as well as in the present one, tractability results for high-dimensional integration are heavily used in the analysis and for the construction of algorithms. For the study of tractability we refer in particular to the monograph series [14, 15]. Furthermore, in [1, 5] as well as in the present paper the multi-level methodology, which was introduced by [4] in the context of integral equations and by [2] in the context of stochastic differential equations, plays a key role.

In the present paper we study integration on the sequence space $\mathbb{R}^{\mathbb{N}}$, as we wish to compute the expectation

$$I(f) = \mathbb{E}(f(X_1, X_2, \ldots))$$

for functions $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, where $(X_1, X_2, \ldots)$ is an i.i.d. sequence of random variables with a common distribution ρ on a Borel subset $D \subseteq \mathbb{R}$. Important examples include the uniform distribution ρ on $[0, 1]$ and the standard normal distribution on the real line. In a reasonable approach to compute the integral $I(f)$ all but finitely many random variables X_j are replaced by some nominal value c of the distribution ρ, and the aim is to construct deterministic quadrature formulas based on a finite number of function values $f(x_{i,1}, \ldots, x_{i,d_i}, c, c, \ldots)$.

Let $\Psi_{1:d} f$ denote the corresponding function of the first d variables, i.e.,

$$(\Psi_{1:d} f)(\mathbf{x}) = f(x_1, \ldots, x_d, c, c \ldots)$$

for $d \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. For example, an equal weight quadrature formula that uses n function values in a fixed dimension d takes the form

$$Q(f) = \frac{1}{n} \sum_{i=1}^{n} (\Psi_{1:d} f)(\mathbf{x}_i) = \frac{1}{n} \sum_{i=1}^{n} f\left(x_{i,1}, \ldots, x_{i,d}, c, c, \ldots\right)$$

for some design of points $(x_{i,1}, \ldots, x_{i,d}) \in \mathbb{R}^d$. The cost of a single function evaluation is assumed to be given by d^s for some $s \geq 0$, so that the cost of

the quadrature formula Q is given by

$$N = n \cdot d^s,$$

which corresponds to the fixed subspace sampling model of [1]. Clearly, $s = 0$ is unrealistic in practical applications, while $s = 1$ is a reasonable choice in many situations.

For quadrature on the sequence space, a multi-level algorithm is based on an increasing sequence $d_1 < \cdots < d_L$ of dimensions, and functions f of infinitely many variables are decomposed as

$$f = \Psi_{1:d_1} f + \sum_{\ell=2}^{L} (\Psi_{1:d_\ell} f - \Psi_{1:d_{\ell-1}} f) + f - \Psi_{1:d_L} f. \tag{1}$$

The function $\Psi_{1:d_1} f$ as well as the functions $\Psi_{1:d_\ell} f - \Psi_{1:d_{\ell-1}} f$ are integrated separately by means of suitable quadrature formulas $Q_1, \ldots, Q_L$ in dimensions $d_1, \ldots, d_L$. This leads to the multi-level algorithm

$$Q(f) = Q_1(\Psi_{1:d_1} f) + \sum_{\ell=2}^{L} \big(Q_\ell(\Psi_{1:d_\ell} f) - Q_\ell(\Psi_{1:d_{\ell-1}} f)\big).$$

Let n_ℓ denote the number of knots used by Q_ℓ. As previously, the cost of evaluating a function of d variables is assumed to be d^s, so that the cost of the multi-level algorithm Q is given by

$$N = n_1 \cdot d_1^s + \sum_{\ell=2}^{L} n_\ell \cdot (d_\ell^s + d_{\ell-1}^s),$$

which corresponds to the varying subspace sampling model of [1]. We refer to [10] for a more generous cost model, where the cost of a function evaluation at a point $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ may depend in any way on the number of components of $\mathbf{x}$ that are different from the nominal value c. Obviously the cost for evaluating $\Psi_{1:d_\ell} f - \Psi_{1:d_{\ell-1}} f$ increases with the level ℓ. Suppose, on the other hand, that these differences get small in a suitable norm, which makes the integration problem easier with increasing ℓ. Then the multi-level decomposition allows to balance these two effects by sampling more frequently in smaller dimensions.

The class F of integrands f that will be studied in this paper is the unit ball $B(K)$ in a Hilbert space $H(K)$ with reproducing kernel K. The construction of K is based on a reproducing kernel k for real-valued functions

of a single variable $x \in D$ and a sequence of weights $\gamma_j > 0$. By assumption, $\int_D k(x,x)\,\rho(dx) < \infty$ and $k(c,c) = 0$, the latter being called the anchored case in the literature, and furthermore we assume $\sum_{j=1}^{\infty} \gamma_j < \infty$. The reproducing kernel K for functions of infinitely many variables is given by

$$K(\mathbf{x},\mathbf{y}) = \sum_{u} \prod_{j\in u} \gamma_j\, k(x_j, y_j),$$

where u varies over all finite subsets of $\mathbb{N}$ and $\mathbf{x}$ and $\mathbf{y}$ belong to a subset of $D^{\mathbb{N}}$ of measure one. Function spaces $H(K)$ of this kind have already been studied in [12, 5, 10, 9]. In particular, (1) is an orthogonal decomposition of $f \in H(K)$.

We study the minimal worst case errors $e_s^{\mathrm{fix}}(N, B(K))$ and $e_s^{\mathrm{var}}(N, B(K))$ that can be achieved by deterministic algorithms that use fixed or variable subspace sampling, respectively, with worst case cost at most N. We derive upper and lower bounds for these quantities, which depend on the decay of the weights γ_j and on respective upper and lower bounds for finite-dimensional integration on the unit balls $B(K_{1:d})$, where

$$K_{1:d}(\mathbf{x},\mathbf{y}) = \sum_{u\subseteq\{1,\dots,d\}} \prod_{j\in u} \gamma_j\, k(x_j, y_j)$$

is a reproducing kernel for functions of the variables $x_1, \dots, x_d \in D$. The upper bounds for $e_s^{\mathrm{var}}(N, B(K))$ are achieved by suitable multi-level algorithms, and in the corresponding analysis auxiliary weights γ_j' such that $\lim_{j\to\infty} \gamma_j/\gamma_j' = 0$ are employed. We refer to [5] for a counterpart of this approach in the analysis of randomized (Monte Carlo) multi-level algorithms.

To give a flavor of our results, consider the uniform distribution ρ on $D = [0,1]$ and the kernel

$$k(x,y) = \min(x,y), \qquad x, y \in [0,1],$$

and assume $\gamma_j = j^{-1-2q}$ with $q > 0$. Furthermore, let $s = 1$ in the definition of the cost. In order to simplify the presentation we put

$$\lambda^{\mathrm{var}} = \sup\{\chi > 0 : \sup_{N\in\mathbb{N}} e_1^{\mathrm{var}}(N, B(K)) \cdot N^{\chi} < \infty\},$$

and we use λ^{fix} to denote the corresponding quantity for fixed subspace sampling. Roughly speaking, λ^{var} and λ^{fix} are the best orders of convergence

that can be achieved by any sequence of algorithms using variable or fixed subspace sampling, respectively. Clearly, $\lambda^{\mathrm{var}} \geq \lambda^{\mathrm{fix}}$. We have

$$\lambda^{\mathrm{fix}} = \frac{q}{q+1}$$

if $q \geq 1/2$, and otherwise the rather tight bounds

$$\frac{q}{2q/(2q+1)+1} \leq \lambda^{\mathrm{fix}} \leq \frac{q}{q+1}$$

hold true. For variable subspace sampling

$$\lambda^{\mathrm{var}} = \min(q, 1)$$

if $|q-1| \geq 1/2$, while otherwise we only know that

$$\frac{q+1/2}{2} \leq \lambda^{\mathrm{var}} \leq \min(q, 1)$$

with a significant gap between the upper and the lower bound. Still we conclude that variable subspace sampling is superior to fixed subspace sampling for all $q > 0$. Moreover, the lower bounds for λ^{var} are achieved by multi-level algorithms based on rank-1 lattice rules, so that we have optimality for these multi-level algorithms in the case $|q-1| \geq 1/2$. The proof of the lower bound for λ^{var} relies on a tractability result from [8] for rank-1 lattice rules.

This article is organized as follows. The present section concludes by describing an application that motivates the problem of computing the expectation of $f(X_1, X_2, \ldots)$. Section 2 defines the Hilbert space $H(K)$ where the function f resides, and appropriate assumptions are made to facilitate the worst case error analysis later. The worst case setting for quadrature on the sequence space and in particular the different cost models are presented in Section 3. In Section 4 key results on the single level algorithm from [12] are reviewed, and upper bounds for the error of multi-level algorithms are derived. Lower bounds are established in Section 5, and Section 6 contains an application of our results for ρ being the uniform distribution on $D = [0, 1]$ and $k = \min$.

For motivation we now consider the problem of option pricing in mathematical finance, which very well fits into the setting of the present paper, see [3]. The option pricing problem amounts to the computation of the expectation $\mathbb{E}(\varphi(S))$, where $S = (S(t))_{t \in [0,T]}$ denotes the asset price over some

time interval $[0,T]$ and φ denotes a discounted and possibly path dependent payoff. Typically, S is modeled by a stochastic differential equation, and here we consider the scalar case for convenience. Thus

$$dS(t) = r(t, S(t))\,dt + \sigma(S(t), t)\,dB(t),$$

with deterministic initial price $S(0) = s_0$ and a scalar Brownian motion B. Consequently, under regularity assumptions on the drift coefficient r and the diffusion coefficient σ, we have $S = \Gamma(B)$ for some measurable mapping $\Gamma : C([0,T]) \to C([0,T])$. For example,

$$\Gamma(u)(t) = s_0 \exp((r - \sigma^2/2)t + \sigma u(t))$$

and

$$\varphi(v) = e^{-rT} \max\left(T^{-1} \int_0^T v(t)\,dt - K, 0\right)$$

in the case of an arithmetic Asian call option with strike K in a Black Scholes model with constant interest rate r and constant volatility σ. Take a series expansion

$$B = \sum_{j=1}^{\infty} X_j \cdot e_j$$

of the Brownian motion with an i.i.d. sequence of standard normal random variables X_j and a sequence of functions $e_j \in C([0,T])$. Then

$$\mathbb{E}(\varphi(S)) = I(f)$$

with $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ given by

$$f(\mathbf{x}) = \varphi \circ \Gamma\left(\sum_{j=1}^{\infty} x_j \cdot e_j\right)$$

and ρ being the standard normal distribution.

Possible choices of of basis functions e_j are provided by the Karhunen-Loéve expansion and by the Lévy-Ciesielski expansion, which is also known as the Brownian bridge construction of B. In the Karhunen-Loéve expansion the basis functions are orthogonal in $L_2([0,T])$ and given by

$$e_j(t) = \sqrt{2T} \cdot \frac{\sin\left((j - 1/2)\pi t/T\right)}{(j - 1/2)\pi}.$$

In the Lévy-Ciesielski construction the basis functions are the Schauder functions and (with a more convenient index set) given by $e_{1,0}(t) = t$ and

$$e_{k,m}(t) = \int_0^t 2^{(m-1)/2} \cdot \big(1_{[(k-1)/2^m, k/2^m[} - 1_{[k/2^m, (k+1)/2^m[}\big)(u)\, du$$

for $k \in \{2j-1 : j = 1, \ldots, 2^{m-1}\}$ and $m \in \mathbb{N}$.

Multi-level algorithms on the sequence space $\mathbb{R}^{\mathbb{N}}$ are applicable if $\varphi \circ \Gamma$ (or a reasonable approximation thereof) can be evaluated at any function $\sum_{j=1}^{d} x_j \cdot e_j$. Results on the smoothness of f within the setting of the present paper seem to be unknown so far.

This multi-level application to problems from computational finance has been suggested and tested in [3] for various options in the Black Scholes model. In the latter paper the inverse of the cumulative normal distribution function is used to transform the problem to an integration problem with ρ being the uniform distribution on $[0, 1]$, and rank-1 lattice rules are used as the building block.

2. The Function Spaces

In this section, the Hilbert space $H(K)$ where the functions f reside is constructed. The elements of $H(K)$ depend on a countably infinite number of variables, and $H(K)$ is constructed as the tensor product space of a countable number of reproducing kernel Hilbert spaces. As discussed in Section 1, the computational problem is to approximate $I(f) = \mathbb{E}(f(X_1, X_2, \ldots))$, where $(X_1, X_2, \ldots)$ is an i.i.d. sequence of random variables with a common distribution ρ on a Borel subset $D \subseteq \mathbb{R}$.

A measurable, symmetric, positive semi-definite kernel function

$$k : D \times D \to \mathbb{R}$$

is the building block used to construct the Hilbert space $H(K)$. This kernel may possibly be unbounded, but we assume that

$$\int_D k(x, x)\, \rho(dx) < \infty. \tag{2}$$

In addition, it is assumed that the nominal value satisfies $c \in D$ and

$$k(c, c) = 0, \tag{3}$$

which implies that $f(c) = 0$ for all functions f in the Hilbert space $H(k)$ with reproducing kernel k.

To facilitate the definition of the Hilbert space $H(K)$ of functions of an infinite number of variables, a sequence $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ of positive weights is introduced, which satisfies the condition

$$\sum_{j=1}^{\infty} \gamma_j < \infty. \tag{4}$$

Under these assumptions the appropriate choice of a domain for the functions f of an infinite number of variables is

$$\mathcal{X} = \Big\{ \mathbf{x} \in D^{\mathbb{N}} : \sum_{j=1}^{\infty} \gamma_j k(x_j, x_j) < \infty \Big\},$$

since

$$(X_1, X_2, \ldots) \in \mathcal{X}$$

holds almost surely, see [5, Lemma 1]. Moreover, $\prod_{j=1}^{\infty}(1 + \gamma_j |k(x_j, y_j)|)$ converges for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, so that

$$K(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{\infty}(1 + \gamma_j k(x_j, y_j))$$

defines a measurable, symmetric, positive semi-definite kernel function

$$K : \mathcal{X} \times \mathcal{X} \to \mathbb{R},$$

see [5, Sec. 2.4]. In the sequel we consider the Hilbert space $H(K)$ with reproducing kernel K.

We discuss the orthogonal decomposition of $f \in H(K)$ into functions that only depend on finitely many variables. Let $\mathbb{U} = \{u \subseteq \mathbb{N} : |u| < \infty\}$ denote the set of all finite subsets of $\mathbb{N}$. For every $u \in \mathbb{U}$ we consider the Hilbert space $H(k_u)$ whose reproducing kernel is given by

$$k_u(\mathbf{x}, \mathbf{y}) = \prod_{j \in u} k(x_j, y_j)$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. By definition, $k_\emptyset = 1$ and therefore $H(k_\emptyset)$ is the space of constant functions. As shown in [9, Sec. 2] in the case of a bounded kernel

k and $D = [0,1]$, the functions $f \in H(k_u)$ satisfy $f(\mathbf{x}) = f(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $x_j = y_j$ for every $j \in u$. Hence $f \in H(k_u)$ may be identified with a function $f : D^u \to \mathbb{R}$ of the variables x_j with $j \in u$ only. Let

$$\gamma_u = \prod_{j \in u} \gamma_j$$

for $u \in \mathbb{U}$ to obtain

$$K(\mathbf{x}, \mathbf{y}) = \sum_{u \in \mathbb{U}} \gamma_u k_u(\mathbf{x}, \mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. We refer to [9] for a proof of the following fact, see also [12, 5].

Lemma 1. *The Hilbert space $H(K)$ consists of all functions*

$$f = \sum_{u \in \mathbb{U}} f_u$$

with $f_u \in H(k_u)$ such that

$$\sum_{u \in \mathbb{U}} \gamma_u^{-1} \|f_u\|_{k_u}^2 < \infty.$$

Moreover, $\langle f, g \rangle_K = \sum_{u \in \mathbb{U}} \gamma_u^{-1} \langle f_u, g_u \rangle_{k_u}$ for $f, g \in H(K)$.

According to Lemma 1, $(H(k_u))_{u \in \mathbb{U}}$ is a family of closed and pairwise orthogonal subspaces of $H(K)$ and f_u is the orthogonal projection of f onto $H(k_u)$. Roughly speaking, f_u represents the joint effect of the variables x_j with $j \in u$ on the function f.

In a reasonable approach to compute the integral $I(f) = \mathbb{E}(f(X_1, X_2, \ldots))$ all but finitely many random variables X_j are replaced by the nominal value c. Hence we define

$$(\Psi_v f)(\mathbf{x}) = f(\mathbf{x}_v, \mathbf{c})$$

for $\mathbf{x} \in \mathcal{X}$ and $v \in \mathbb{U}$, where $(\mathbf{x}_v, \mathbf{c})$ is used to denote the sequence $\mathbf{y} \in \mathcal{X}$ with $y_j = x_j$ for $j \in v$ and $y_j = c$ otherwise. For $f = \sum_{u \in \mathbb{U}} f_u$ according to Lemma 1 we get $f_u(\mathbf{x}_v, \mathbf{c}) = 0$ if $u \not\subseteq v$ because of (3), and therefore

$$\Psi_v f = \sum_{u \subseteq v} f_u.$$

We conclude that $\Psi_v f$ is the orthogonal projection of f onto the Hilbert space $H(K_v)$ with reproducing kernel

$$K_v(\mathbf{x}, \mathbf{y}) = \sum_{u \subseteq v} \gamma_u k_u(\mathbf{x}, \mathbf{y}) = \prod_{j \in v} (1 + \gamma_j k(x_j, y_j)).$$

We add that $f \in H(K_v)$ may be identified with a function $f : D^v \to \mathbb{R}$ of the variables x_j with $j \in v$ only.

3. The Integration Problem

The assumptions (2) and (4) guarantee that $f \mapsto I(f)$ defines a bounded linear function on $H(K)$, see [5, Sec. 2.5]. We approximate I by means of quadrature formulas

$$Q(f) = \sum_{i=1}^{n} a_i f(\mathbf{x}_i)$$

with coefficients $a_i \in \mathbb{R}$ and knots $\mathbf{x}_i \in \mathcal{X}$, whose components coincide with the nominal value c for all but finitely many coordinates.

We study the worst case error

$$e(Q, F) = \sup_{f \in F} |I(f) - Q(f)|$$

of Q on function classes F, where we are primarily interested in the case of the unit ball

$$B(K) = \{f \in H(K) : \|h\|_K \leq 1\}$$

in $H(K)$.

We study two different cost models for the infinite-dimensional quadrature problem. Let $1 : d = \{1, \ldots, d\}$ and

$$\mathcal{X}_{1:d} = \{\mathbf{x} \in D^{\mathbb{N}} : x_{d+1} = x_{d+2} = \cdots = c\}.$$

Clearly $\mathcal{X}_{1:d} \subseteq \mathcal{X}$, and $\mathcal{X}_{1:d}$ may be considered as a d-dimensional affine subspace of $\mathbb{R}^{\mathbb{N}}$. As the basic assumption in both models, for every dimension d an oracle is available that provides values of f at any knot $\mathbf{x} \in \mathcal{X}_{1:d}$, and the cost of a single function evaluation by means of this oracle is given by d^s with some fixed parameter $s > 0$. In the fixed subspace model every quadrature

formula Q uses a single oracle for some dimension d to provide all function values $f(\mathbf{x}_i)$. Accordingly, the cost $c_s^{\mathrm{fix}}(Q)$ of Q is defined by

$$c_s^{\mathrm{fix}}(Q) = n \cdot (\min\{d \in \mathbb{N} : \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathcal{X}_{1:d}\})^s$$

if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the pairwise different knots of Q. In the variable subspace model a quadrature formula may use oracles for different dimensions. This leads to the definition of the cost

$$c_s^{\mathrm{var}}(Q) = \sum_{i=1}^{n} (\min\{d \in \mathbb{N} : \mathbf{x}_i \in \mathcal{X}_{1:d}\})^s$$

of Q in the variable subspace model.

In [10] a more generous cost model is introduced, where the cost of a function evaluation at $\mathbf{x}$ is linked to the number

$$\ell(\mathbf{x}) = |\{j \in \mathbb{N} : x_j \neq c\}|$$

of components of $\mathbf{x}$ that are different from the nominal value c. In this model the cost of Q is defined by

$$c_s^*(Q) = \sum_{i=1}^{n} (\max(\ell(\mathbf{x}), 1))^s.$$

For a cost budget $N \in \mathbb{N}$ and $\dagger \in \{\mathrm{fix}, \mathrm{var}, ^*\}$ the minimal errors in the corresponding cost models are defined by

$$e_s^\dagger(N, F) = \inf\{e(Q, F) : c_s^\dagger(Q) \leq N\}.$$

Clearly $c_s^*(Q) \leq c_s^{\mathrm{var}}(Q) \leq c_s^{\mathrm{fix}}(Q)$, and therefore

$$e_s^*(N, F) \leq e_s^{\mathrm{var}}(N, F) \leq e_s^{\mathrm{fix}}(N, F).$$

In this paper we study the asymptotic behavior of the minimal errors $e_s^{\mathrm{fix}}(N, B(K))$ and $e^{\mathrm{var}}(N, B(K))$, and we also compare this behavior with results on $e_s^*(N, B(K))$ from [10]. Furthermore, we study the construction of quadrature formulas with cost bounded by N and error close to the corresponding minimal error. In order to simplify the presentation we introduce the exponents

$$\lambda_s^\dagger(B(K)) = \sup\{\chi > 0 : \sup_{N \in \mathbb{N}} e_s^\dagger(N, B(K)) \cdot N^\chi < \infty\} \tag{5}$$

for $\dagger \in \{\mathrm{fix}, \mathrm{var}, ^*\}$.

4. Multi-level Algorithms

In a common approach to the integration problem the infinite-dimensional integral $I(f) = \mathbb{E}(f(X_1, X_2, \ldots))$ is approximated by a finite-dimensional integral

$$I(\Psi_{1:d} f) = \mathbb{E}(f(X_1, \ldots, X_d, c, c, \ldots)),$$

and a quadrature formula Q for integration with respect to the joint distribution of $X_1, \ldots, X_d$, i.e., with respect to the d-fold product of the probability measure ρ, is used to approximately compute $I(\Psi_{1:d} f)$. Hence there are two kinds of errors, namely $|I(f) - I(\Psi_{1:d} f)|$, which is due to the truncation of the infinite-dimensional integral, and $|I(\Psi_{1:d} f) - Q(f)|$ for the d-dimensional integration problem. Recall that $\Psi_{1:d} f$ is the orthogonal projection of f onto $H(K_{1:d})$. Under the natural assumption that Q uses knots from the space $\mathcal{X}_{1:d}$ we have $Q(f) = Q(\Psi_{1:d} f)$. Then the decomposition of the error corresponds to the orthogonal decomposition of $H(K)$ into the space $H(K_{1:d})$ and its complement.

4.1. The Multi-level Construction

In the multi-level approach the space $H(K_{1:d})$ is further decomposed as follows. Consider a sequence of increasing dimensions

$$1 \leq d_1 < \cdots < d_L = d$$

with the associated closed subspaces

$$H(K_{1:d_1}) \subseteq \cdots \subseteq H(K_{1:d_L})$$

of $H(K)$. For $f \in H(K)$ we employ the orthogonal decomposition

$$f = \Psi_{1:d_1} f + \sum_{\ell=2}^{L} (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}}) f + f - \Psi_{1:d_L} f. \tag{6}$$

Roughly speaking, $(\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}}) f$ yields the part of f that depends on the first d_ℓ variables, but not only on the first $d_{\ell-1}$ variables. For integration of these parts we choose quadrature formulas $Q_1, \ldots, Q_L$, and we apply the so-called multi-level algorithm

$$Q = Q_1 \circ \Psi_{1:d_1} + \sum_{\ell=2}^{L} Q_\ell \circ (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})$$

for integration of $\Psi_{1:d_L} f$.

Suppose that

$$Q_\ell(f) = \sum_{i=1}^{n_\ell} a_i^{(\ell)} f(\mathbf{x}_i^{(\ell)})$$

with coefficients $a_i^{(\ell)} \in \mathbb{R}$ and knots $\mathbf{x}_i^{(\ell)} \in \mathcal{X}$. Then the quadrature formulas $Q_\ell \circ \Psi_{1:d_\ell}$ and $Q_\ell \circ \Psi_{1:d_{\ell-1}}$ for $\ell > 1$ use the coefficients $a_i^{(\ell)} \in \mathbb{R}$ together with the knots

$$((\mathbf{x}_i^{(\ell)})_{1:d_\ell}, \mathbf{c}) \in \mathcal{X}_{1:d_\ell}$$

and

$$((\mathbf{x}_i^{(\ell)})_{1:d_{\ell-1}}, \mathbf{c}) \in \mathcal{X}_{1:d_{\ell-1}},$$

respectively. In particular,

$$c_s^{\mathrm{var}}(Q) \le n_1 \cdot d_1^s + \sum_{\ell=2}^{L} n_\ell \cdot (d_\ell^s + d_{\ell-1}^s) \le 2 \cdot \sum_{\ell=1}^{L} n_\ell \cdot d_\ell^s \tag{7}$$

holds for the cost of the multi-level algorithm Q in the variable subspace model.

4.2. General Error Bounds

Now we turn to the error analysis of multi-level algorithms. Put

$$b_{1:d}(B(K)) = \sup_{f \in B(K)} |I(f) - I(\Psi_{1:d} f)|,$$

which is the worst case truncation error, and let $B(K_{1:d})$ denote the unit ball in $H(K_{1:d})$.

Theorem 1. *Under the assumptions* (2), (3), *and* (4) *the error of the multi-level algorithm Q satisfies*

$$\begin{aligned} &e^2(Q, B(K)) \\ &\quad = e^2(Q_1, B(K_{1:d_1})) + \sum_{\ell=2}^{L} \big(e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}})) \big) \\ &\qquad + b_{1:d_L}^2(B(K)). \end{aligned}$$

Proof. Let h denote the representer of I, and let g_ℓ denote the representer of Q_ℓ, i.e.,

$$I(f) = \langle f, h \rangle_K$$

and

$$Q_\ell(f) = \langle f, g_\ell \rangle_K$$

for every $f \in H(K)$. Then the representer g of Q is given by

$$g = \Psi_{1:d_1} g_1 + \sum_{\ell=2}^{L} (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}}) g_\ell \in H(K_{1:d_L}),$$

which implies $\Psi_{1:d_L} g = g$. Use the orthogonal decomposition (6) for $f = h - g$ to obtain

$$\begin{aligned} e^2(Q, B(K)) &= \|h - g\|_K^2 \\ &= \|\Psi_{1:d_1}(h - g_1)\|_K^2 + \sum_{\ell=2}^{L} \|(\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})(h - g_\ell)\|_K^2 \\ &\quad + \|h - \Psi_{1:d_L} h\|_K^2. \end{aligned}$$

Moreover, for $\ell > 1$,

$$\begin{aligned} &\|(\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})(h - g_\ell)\|_K^2 \\ &\quad = \|\Psi_{1:d_\ell}(h - g_\ell)\|_K^2 - \|\Psi_{1:d_{\ell-1}}(h - g_\ell)\|_K^2 \\ &\quad = e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}})), \end{aligned}$$

and

$$\|\Psi_{1:d_1}(h - g_1)\|_K^2 = e^2(Q_1, B(K_{1:d_1})).$$

□

According to Theorem 1 the squared error of the multi-level algorithm Q can be decomposed into its squared truncation error and differences of squared errors of the quadrature formulas Q_ℓ for integration in dimensions d_ℓ and $d_{\ell-1}$. Note that these differences are always non-negative.

Remark 1. In the particular case $L = 1$, i.e., for a single-level algorithm, Theorem 1 yields the decomposition

$$e^2(Q, B(K)) = b_{1:d}^2(B(K)) + e^2(Q, B(K_{1:d})), \tag{8}$$

which is due to [12, Thm. 1]. See [5, Lemma 8] for a counterpart for randomized algorithms.

Put

$$m = \int_D \int_D k(x,y)\, \rho(dx)\, \rho(dy)$$

and note that (2) implies $m < \infty$. Hence

$$b^2_{1:d}(B(K)) = \prod_{j=1}^{d}(1+\gamma_j m)\left(\prod_{j=d+1}^{\infty}(1+\gamma_j m) - 1\right) \preceq \sum_{j=d+1}^{\infty} \gamma_j \qquad (9)$$

holds for the truncation error, see [12].

Remark 2. In the particular case of equal weight quadrature formulas

$$Q_\ell(f) = \frac{1}{n_\ell}\sum_{i=1}^{n_\ell} f(\mathbf{x}_i^{(\ell)})$$

the term $e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}}))$ is the the difference between square discrepancies of the same design $\{\mathbf{x}_1^{(\ell)}, \ldots, \mathbf{x}_{n_\ell}^{(\ell)}\}$ with respect to the kernel functions $K_{1:d_\ell}$ and $K_{1:d_{\ell-1}}$.

Our overall goal is to minimize the error of multi-level algorithms Q subject to a cost bound $c_s^{\mathrm{var}}(Q) \le N$. While Theorem 1 provides an explicit representation of the error, it is technically difficult to directly work with the differences of errors for finite-dimensional integration problems. Thus we are interested in useful upper bounds for these differences. The trivial bound

$$e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}})) \le e^2(Q_\ell, B(K_{1:d_\ell}))$$

immediately removes any advantage of the multi-level approach, but a modification of this idea works well. To this end we introduce a suitable kernel function K', which induces a weaker norm than K, such that

$$\|(\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})f\|_{K'} \le \kappa_\ell^{1/2} \cdot \|(\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})f\|_K$$

with suitable numbers $\kappa_\ell \le 1$. For randomized algorithms a counterpart of this approach was developed in [5].

For the construction of K' we consider another sequence $\boldsymbol{\gamma}' = (\gamma'_1, \gamma'_2, \ldots)$ of positive weights, which satisfies the conditions

$$\gamma_j \le \gamma'_j \qquad (10)$$

for every $j \in \mathbb{N}$ and

$$\sum_{j=1}^{\infty} \gamma'_j < \infty. \tag{11}$$

We use these new weights to define

$$K'(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{\infty} (1 + \gamma'_j k(x_j, y_j))$$

for $\mathbf{x}, \mathbf{y} \in \mathcal{X}'$ with

$$\mathcal{X}' = \Big\{ \mathbf{x} \in D^{\mathbb{N}} : \sum_{j=1}^{\infty} \gamma'_j k(x_j, x_j) < \infty \Big\}.$$

The kernel functions K and K' differ only in terms of the different weights used to define them. Hence Lemma 1 applies to K' as well, and the projections Ψ_v act on $H(K')$ in the same way as they act on $H(K)$.

Let γ'_u etc. be defined in the canonical way. From (10) we get $\gamma_u \leq \gamma'_u$ as well as $\mathcal{X}' \subseteq \mathcal{X}$. If $\mathcal{X}' = \mathcal{X}$ then $H(K) \subseteq H(K')$. In general,

$$if = f|_{\mathcal{X}'}$$

defines a bounded linear mapping $i : H(K) \to H(K')$. Furthermore, we may identify the sets $H(k_v)$ and $H(k'_v)$ as well as $H(K_v)$ and $H(K'_v)$, since their elements only depend on the variables x_j with $j \in v$.

Theorem 2. *Put $\kappa_1 = 1$ and let*

$$\kappa_\ell = \max_{d_{\ell-1} < j \leq d_\ell} \frac{\gamma_j}{\gamma'_j}$$

for $2 \leq \ell \leq L$. Under the assumptions (2), (3), (10), *and* (11) *the error of the multi-level algorithm Q satisfies*

$$e^2(Q, B(K)) \leq b^2_{1:d_L}(B(K)) + \sum_{\ell=1}^{L} \kappa_\ell \cdot e^2(Q_\ell, B(K'_{1:d_\ell})).$$

Proof. In view of Theorem 1 is suffices to show that

$$e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}})) \leq \kappa_\ell \cdot e^2(Q_\ell, B(K'_{1:d_\ell}))$$

for $\ell > 1$ and

$$e^2(Q_1, B(K_{1:d_1})) \le e^2(Q_1, B(K'_{1:d_1})).$$

At first we determine the adjoint i^* of i. Let $f' = \sum_{u\in\mathbb{U}} f'_u \in H(K')$ and $g = \sum_{u\in\mathbb{U}} g_u \in H(K)$ according to Lemma 1. Then

$$f = \sum_{u\in\mathbb{U}} \frac{\gamma_u}{\gamma'_u} f'_u \in H(K)$$

and

$$\langle f, g\rangle_K = \sum_{u\in\mathbb{U}} \gamma_u^{-1} \left\langle \frac{\gamma_u}{\gamma'_u} f'_u, g_u \right\rangle_{k_u} = \sum_{u\in\mathbb{U}} (\gamma'_u)^{-1} \langle f'_u, g_u\rangle_{k_u} = \langle f', g\rangle_{K'},$$

i.e., $i^*(f') = f$.

Put

$$\mathbb{U}_1 = \{u \in \mathbb{U} : u \subseteq 1 : d_1\}$$

and

$$\mathbb{U}_\ell = \{u \in \mathbb{U} : u \subseteq 1 : d_\ell \text{ and } u \not\subseteq 1 : d_{\ell-1}\}$$

for $\ell > 1$. If $f'_u = 0$ for every $u \notin \mathbb{U}_\ell$, then

$$\begin{aligned} \|i^* f'\|_K^2 = \langle f', ii^* f'\rangle_{K'} &= \sum_{u\in\mathbb{U}_\ell} \frac{\gamma_u}{\gamma'_u} (\gamma'_u)^{-1} \langle f'_u, f'_u\rangle_{k_u} \\ &\le \max_{u\in\mathbb{U}_\ell} \frac{\gamma_u}{\gamma'_u} \|f'\|_{K'}^2 \le \kappa_\ell \|f'\|_{K'}^2 . \end{aligned} \tag{12}$$

Now we use the notation and facts from the proof of Theorem 1. Let h' and g'_ℓ denote the representers of I and Q_ℓ, respectively, on the space $H(K')$. Put

$$f'_\ell = (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})(h' - g'_\ell)$$

and

$$f_\ell = (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}})(h - g_\ell)$$

for $\ell > 1$ as well as

$$f'_1 = \Psi_{1:d_1}(h' - g'_1)$$

and

$$f_1 = \Psi_{1:d_1}(h - g_1).$$

Note that f'_ℓ and f_ℓ represent the same functional, and therefore $i^* f'_\ell = f_\ell$ for $\ell = 1, \dots, L$. From (12) we get

$$\begin{aligned} e^2(Q_\ell, B(K_{1:d_\ell})) - e^2(Q_\ell, B(K_{1:d_{\ell-1}})) &= \|f_\ell\|_K^2 \\ &\le \kappa_\ell \cdot \|f'_\ell\|_{K'}^2 = \kappa_\ell \cdot \left(e^2(Q_\ell, B(K'_{1:d_\ell})) - e^2(Q_\ell, B(K'_{1:d_{\ell-1}}))\right) \\ &\le \kappa_\ell \cdot e^2(Q_\ell, B(K'_{1:d_\ell})) \end{aligned}$$

if $\ell > 1$, as claimed. Analogously,

$$e^2(Q_1, B(K_{1:d_1})) = \|f_1\|_K^2 \le \|f'_1\|_{K'}^2 = e^2(Q_\ell, B(K'_{1:d_\ell})).$$

□

The advantage of introducing the new set of weights γ'_j is an upper bound of a simpler form in Theorem 2, which is suitable for optimization under the cost constraint $c_s^{\mathrm{var}}(Q) \le N$. The disadvantage is that the upper bound is not necessarily tight. The next section explores the choice of γ'_j, L, d_ℓ, n_ℓ, and Q_ℓ to make the upper bound as small as possible under the given cost constraint.

4.3. Error Bounds under Strong Tractability Assumptions

In the sequel we strengthen our assumptions on the sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ of weights as well as on the reproducing kernel k. Concerning the weights we assume that

$$\gamma_j \preceq j^{-1-2q} \tag{13}$$

and

$$\gamma'_j = j^{2(q-q')} \cdot \gamma_j \tag{14}$$

with

$$q \ge q' > 0,$$

which implies (4), (10), and (11). Furthermore,

$$b_{1:d}^2(B(K)) \preceq d^{-2q} \tag{15}$$

follows from (9) and (13).

For the finite-dimensional integration problems on the unit balls $B(K'_{1:d})$ we assume strong tractability, namely,

$$\sup_{d \in \mathbb{N}} \inf\{e(Q, B(K'_{1:d})) : Q \text{ } n\text{-point quadrature formula}\} \preceq n^{-p'} \tag{16}$$

with $p' > 0$, see, e.g., [8, 13, 16, 17, 6] as well as Section 6. Note that p' typically depends on q' and the kernel k.

For a vector $\mathbf{d} \in \mathbb{N}^L$ of increasing dimensions $d_1 \le \ldots \le d_L$ and a vector $\mathbf{n} \in \mathbb{N}^L$ of integers $n_1, \ldots, n_L$ we put

$$\mathcal{U}^2(\mathbf{n}, \mathbf{d}) = d_L^{-2q} + \sum_{\ell=1}^{L} n_\ell^{-2p'} \cdot d_{\ell-1}^{-2(q-q')},$$

where $d_0 = 1$, as well as

$$\mathcal{C}(\mathbf{n}, \mathbf{d}) = \sum_{\ell=1}^{L} n_\ell \cdot d_\ell^s.$$

Lemma 2. *Assume* (2), (3), (13), (14), *and* (16). *For every $L \in \mathbb{N}$ and all vectors $\mathbf{n} \in \mathbb{N}^L$ of integers and all vectors $\mathbf{d} \in \mathbb{N}^L$ of increasing dimensions there exists a multi-level algorithm $Q_{\mathbf{n},\mathbf{d}}$ such that*

$$e(Q_{\mathbf{n},\mathbf{d}}, B(K)) \preceq \mathcal{U}(\mathbf{n}, \mathbf{d})$$

and

$$c_s^{\mathrm{var}}(Q_{\mathbf{n},\mathbf{d}}) \preceq \mathcal{C}(\mathbf{n}, \mathbf{d}).$$

Proof. Due to assumption (16) there exist quadrature formulas Q_{n_ℓ,d_ℓ}, which use n_ℓ knots from the space $\mathcal{X}_{1:d_\ell}$ and satisfy

$$e(Q_{n_\ell,d_\ell}, B(K'_{1:d_\ell})) \preceq n_\ell^{-p'}$$

for $\ell = 1, \ldots, L$. Consider the multi-level algorithm

$$Q_{\mathbf{n},\mathbf{d}} = Q_{n_1,d_1} \circ \Psi_{1:d_1} + \sum_{\ell=2}^{L} Q_{n_\ell,d_\ell} \circ (\Psi_{1:d_\ell} - \Psi_{1:d_{\ell-1}}).$$

Clearly, (7) yields the cost bound for $Q_{\mathbf{n},\mathbf{d}}$. Furthermore, (13) and (14) imply

$$\kappa_\ell = \max_{d_{\ell-1} < j \le d_\ell} j^{-2(q-q')} = d_{\ell-1}^{-2(q-q')}$$

for $\ell > 1$. Observe (15) and apply Theorem 2 to obtain the error bound for $Q_{\mathbf{n},\mathbf{d}}$. □

We minimize the upper error bound $\mathcal{U}(\mathbf{n}, \mathbf{d})$ under the cost constraint $\mathcal{C}(\mathbf{n}, \mathbf{d}) \leq N$, which leads to an upper bound for the minimal error for variable subspace sampling. The result depends on the parameters q and q', which control the decay of the weights γ_j and γ'_j, on the exponent p' in the tractability assumption (16), and on the exponent s, which controls the cost of a single function evaluation.

Theorem 3. *Assume* (2), (3), (13), (14), *and* (16). *Then the minimal errors for variable subspace sampling satisfy*

$$e_s^{\mathrm{var}}(N, B(K)) \preceq \begin{cases} N^{-p' \cdot \min\left(1, \frac{q}{p's+q'}\right)} & \text{if } p's + q' \neq q, \\ N^{-p'} \cdot (\log_2 N)^{p'+1/2} & \text{if } p's + q' = q. \end{cases}$$

Proof. According to Lemma 2

$$e_s^{\mathrm{var}}(N, B(K)) \leq \mathcal{U}_N, \tag{17}$$

where

$$\mathcal{U}_N = \inf\{\mathcal{U}(\mathbf{n}, \mathbf{d}) : \mathcal{C}(\mathbf{n}, \mathbf{d}) \leq N\}.$$

Hence it remains to establish suitable upper bounds for $\mathcal{U}_N$.

Put

$$\eta = q/(p's + q'),$$

and choose

$$\xi \in \begin{cases}]p'/(q-q'), 1/s[& \text{if } \eta > 1, \\ \{1/s\} & \text{if } \eta = 1, \\ ]p'/(p's+q'), p'/q[& \text{if } \eta < 1. \end{cases}$$

Moreover, put

$$\beta = \begin{cases} 1 & \text{if } \eta \geq 1, \\ 1 - \xi q'/p' & \text{otherwise.} \end{cases}$$

We define

$$L = \begin{cases} \lceil \log_2 N \rceil & \text{if } \eta > 1, \\ \lceil \log_2(N/\log_2 N) \rceil & \text{if } \eta = 1, \\ \lceil \log_2 N/(\xi \cdot (p's+q')/p') \rceil & \text{if } \eta < 1, \end{cases} \tag{18}$$

and we define $\mathbf{n}^{(N)} \in \mathbb{N}^L$ and $\mathbf{d}^{(N)} \in \mathbb{N}^L$ by

$$n_\ell = \left\lceil 2^{L-\beta\cdot\ell} \right\rceil, \qquad d_\ell = \left\lceil 2^{\xi\cdot\ell} \right\rceil \tag{19}$$

for $\ell = 1, \ldots, L$.

By definition,

$$\mathcal{C}(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp \sum_{\ell=1}^{L} 2^{L-\beta\cdot\ell} \cdot 2^{s\xi\cdot\ell} = 2^{L} \cdot \sum_{\ell=1}^{L} 2^{(s\xi-\beta)\cdot\ell}$$

and

$$\begin{aligned}\mathcal{U}^2(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) &\asymp 2^{-2L\xi q} + \sum_{\ell=1}^{L} 2^{-2p'\cdot(L-\beta\cdot\ell)} \cdot 2^{-2(q-q')\xi\cdot\ell} \\ &\asymp 2^{-2L\xi q} + 2^{-2Lp'} \cdot \sum_{\ell=1}^{L} 2^{-2((q-q')\xi-p'\beta)\cdot\ell}.\end{aligned}$$

Assume that $\eta > 1$. Then $s\xi - \beta = s\xi - 1 < 0$ and therefore

$$\mathcal{C}(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp 2^{L} \asymp N.$$

Furthermore, we have $(q - q')\xi - p'\beta = (q - q')\xi - p' > 0$, which implies $q\xi > p'$ and consequently

$$\mathcal{U}^2(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp 2^{-2L\xi q} + 2^{-2Lp'} \asymp 2^{-2Lp'} \asymp N^{-2p'}.$$

Next, consider the case $\eta = 1$. Then $s\xi - \beta = (q - q')\xi - p'\beta = 0$, which yields

$$\mathcal{C}(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp 2^{L} \cdot L \asymp (N/\log_2 N) \cdot \log_2(N/\log_2 N) \asymp N$$

as well as

$$\begin{aligned}\mathcal{U}^2(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) &\asymp 2^{-2L\xi q} + 2^{-2Lp'} \cdot L \asymp 2^{-2Lp'} \cdot L \\ &\asymp (N/\log_2 N)^{-2p'} \cdot \log_2(N/\log_2 N) \asymp N^{-2p'} \cdot (\log_2 N)^{2p'+1}.\end{aligned}$$

Finally, assume that $\eta < 1$. Then $s\xi - \beta = \xi(p's + q')/p' - 1 > 0$, which implies

$$\mathcal{C}(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp 2^{L} \cdot 2^{(s\xi-\beta)\cdot L} = 2^{L\cdot\xi(p's+q')/p'} \asymp N.$$

Moreover, we have $(q - q')\xi - p'\beta = \xi q - p' < 0$ and

$$(\log_2 N)p'\eta \leq L\xi q \leq (\log_2 N)p'\eta + \xi q,$$

which leads to

$$\mathcal{U}^2(\mathbf{n}^{(N)}, \mathbf{d}^{(N)}) \asymp 2^{-2L\xi q} + 2^{-2Lp'} \cdot 2^{2(p'-\xi q)\cdot L} \asymp 2^{-2L\xi q} \asymp N^{-2p'\eta}.$$

□

Remark 3. The proof of Theorem 3 is constructive in the following sense. Assume that $\gamma'_j \preceq j^{-1-2q'}$ permits the construction of n-point quadrature formulas $Q_{n,d}$ for all $n, d \in \mathbb{N}$, such that $Q_{n,d}$ uses knots from the space $\mathcal{X}_{1:d}$ and

$$\sup_{d\in\mathbb{N}} e(Q_{n,d}, B(K'_{1:d})) \preceq n^{-p'},$$

cf. (16). Then the multi-level algorithms defined by (18) and (19) yield the upper bound for $e_s^{\mathrm{var}}(N, B(K))$ in Theorem 3. This convergence rate is realized by focusing more sampling effort on the lower dimensions.

Remark 4. We stress that (17) is only an upper bound for the minimal error $e_s^{\mathrm{var}}(N, B(K))$, since in its derivation we have imposed a multi-level structure of the quadrature formulas, which effects the analysis of the cost, and we have employed auxiliary weights, which effects the analysis of the error. Nevertheless, we add that the upper bound for $\mathcal{U}_N$ from the proof of Theorem 3 is sharp, at least if $q \neq p's + q'$.

For a proof of this fact let $\mathbf{d} \in \mathbb{N}^L$ with $d_1 \leq \ldots \leq d_L$ and $\mathbf{n} \in \mathbb{N}^L$ such that $\mathcal{C}(\mathbf{n}, \mathbf{d}) \leq N$. Then $n_1 \leq N$ and $n_L d_{L-1}^s \leq n_L d_L^s \leq N$, and therefore

$$\begin{aligned}
\mathcal{U}(\mathbf{n}, \mathbf{d}) &\geq n_1^{-2p'} + n_L^{-2p'} \cdot d_{L-1}^{-2(q-q')} + d_L^{-2q} \\
&\geq N^{-2p'} + n_L^{-2p'} (N/n_L)^{-2(q-q')/s} + (N/n_L)^{-2q/s} \\
&= N^{-2p'} + N^{-2p'\eta} \cdot \left(r^{2q(1-1/\eta)/s} + r^{2q/s}\right)
\end{aligned}$$

with

$$r = n_L / N^{q'\eta/q}.$$

If $\eta < 1$, then $r^{2q(1-1/\eta)/s} + r^{2q/s} \geq 1$ and therefore $\mathcal{U}(\mathbf{n}, \mathbf{d}) \geq N^{-2p'\eta}$. In any case, $\mathcal{U}(\mathbf{n}, \mathbf{d}) \geq N^{-2p'}$.

Together with the upper bound from the proof of Theorem 3 this yields

$$\mathcal{U}_N \asymp N^{-2p' \cdot \min\left(1, \frac{q}{p's+q'}\right)}$$

if $q \neq p's + q'$. In the case $q = p's + q'$ we conclude that

$$N^{-2p'} \preceq \mathcal{U}_N \preceq N^{-2p'} \cdot (\log_2 N)^{2p'+1}.$$

Theorem 3 yields

$$\lambda_s^{\mathrm{var}}(B(K)) \geq \tau_s^{\mathrm{var}}(q)$$

where $\lambda_s^{\mathrm{var}}(B(K))$ is defined in (5) and

$$\tau_s^{\mathrm{var}}(q) = \sup \min\left(p', \frac{q}{s + q'/p'}\right) \leq \min\left(\sup p', q/s\right). \tag{20}$$

Here the suprema are taken over all $q' \leq q$ and p' such that the strong tractability assumption (16) is satisfied. This optimization is a nontrivial, but sometimes solvable problem. In some cases, even the upper bound in (20) is attainable and $\lambda_s^{\mathrm{var}}(B(K)) = \tau_s^{\mathrm{var}}(q)$, see Section 6.

Remark 5. For single-level algorithms, i.e., in the case $L = 1$, there is no advantage in using auxiliary weights γ_j'. Hence we take $q' = q$ so that (16) becomes a strong tractability assumption for integration on the unit balls $B(K_{1:d})$. Here $p' > 0$ typically depends on q and k. Accordingly

$$\mathcal{U}^2(n, d) = d^{-2q} + n^{-2p'}$$

and

$$\mathcal{C}(n, d) = nd^s.$$

Minimizing $\mathcal{U}^2(n, d)$ with respect to d and n given the constraint $\mathcal{C}(n, d) \leq N$ yields the following result. Under the assumptions (2), (3), (13), and (16) we have

$$\lambda_s^{\mathrm{fix}}(B(K)) \geq \frac{p''}{1 + p''s/q}$$

for p'' being the supremum over all p' such that the strong tractability assumption (16) is satisfied. See [12]. Again the proof is constructive in the sense of Remark 3.

5. Lower Bounds for the Minimal Errors

To derive lower bounds for the minimal errors $e_s^{\mathrm{fix}}(N, B(K))$ as well as $e_s^{\mathrm{var}}(N, B(K))$ for the infinite-dimensional integration problem we consider two extremal cases. Either we only take into account the truncation error and suppose that any finite-dimensional integral can be computed exactly, or we ignore the truncation error and only consider integration with respect to a single variable.

In the latter case we employ the minimal error

$$e(n, B(k)) = \inf\{e(Q, B(k)) : Q \text{ } n\text{-point quadrature formula}\}$$

for integration of functions $f : D \to \mathbb{R}$ from the unit ball $B(k)$ in the Hilbert space $H(k)$. We assume that

$$m = \int_D \int_D k(x,y)\, \rho(dx)\, \rho(dy) > 0, \tag{21}$$

which excludes that $\int_D f(x)\, \rho(dx) = 0$ for all $f \in H(k)$.

Theorem 4. *Under the assumptions* (2), (3), (4), *and* (21) *the minimal errors satisfy*

$$e_s^{\mathrm{fix}}(N, B(K)) \succeq \inf_{n \cdot d^s \le N} \left(\left(\sum_{j=d+1}^{\infty} \gamma_j \right)^{1/2} + e(\lfloor N/d^s \rfloor, B(k)) \right)$$

and

$$e_s^{\mathrm{var}}(N, B(K)) \succeq \left(\sum_{j=\lfloor N^{1/s} \rfloor + 1}^{\infty} \gamma_j \right)^{1/2} + e(N, B(k)).$$

Proof. At first we derive the lower bound for variable subspace sampling. Consider a quadrature rule Q with knots $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}$ such that $c_s^{\mathrm{var}}(Q) \le N$. Then Q formally is a single-level algorithm with $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathcal{X}_{1:d}$ for $d = \lfloor N^{1/s} \rfloor$, and therefore (8) is applicable. Observe that (9) and (21) imply

$$b_{1:d}^2(B(K)) \asymp \sum_{j=d+1}^{\infty} \gamma_j.$$

Finally, $n \le N$ and therefore

$$e(Q, B(K_{1:d})) \ge \gamma_1^{1/2} \cdot e(Q, B(k_{\{1\}})) \ge \gamma_1^{1/2} \cdot e(N, B(k)).$$

For fixed subspace sampling $c_s^{\mathrm{fix}}(Q) \le N$ implies $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathcal{X}_{1:d}$ with $n \cdot d^s \le N$. Now we proceed as previously. □

In [10, Thm. 3] a more sophisticated analysis is used to obtain a lower bound for $e_s^*(N, B(K))$.

6. An Application

We consider the Lebesgue measure ρ on $D = [0,1]$ together with the kernel function

$$k(x,y) = \min(x,y)$$

for $x, y \in D$. Clearly (2) is satisfied, and (3) holds for the nominal value $c = 0$. Furthermore, we assume a matching lower bound in (13), i.e.,

$$\gamma_j \asymp j^{-1-2q}.$$

It is well known that

$$e(n, B(k)) \asymp n^{-1}$$

for the corresponding one-dimensional integration problem.

From Theorem 4 we get the following facts, which correspond to lower bounds for minimal errors.

Corollary 1. *We have*

$$\lambda_s^{\mathrm{fix}}(B(K)) \le \frac{q}{q+s}$$

and

$$\lambda_s^{\mathrm{var}}(B(K)) \le \min\left(\frac{q}{s}, 1\right).$$

Strong tractability results for the corresponding finite-dimensional integration problems with respect to the uniform distribution on $[0,1]^d$ are established in [8, 17], see also [15]. We take auxiliary weights γ_j' according to (14). Since $\gamma_j' \asymp j^{-1-2q'}$ with $q' > 0$, the strong tractability assumption (16) is satisfied for every

$$p' < \min(1, q' + 1/2),$$

and the corresponding n-point quadrature formulas may be chosen as rank-1 lattice rules, see [8].

Now we turn to the optimization problem (20) how to select the parameters of the corresponding multi-level algorithms. In the case

$$0 < q < s/2$$

we choose q' arbitrarily close to zero to obtain p' arbitrarily close to $1/2$ and

$$\tau_s^{\mathrm{var}}(q) = \frac{q}{s}.$$

In the case

$$s/2 \le q < s + 1/2$$

we choose

$$q' = \frac{q - s/2}{s+1}$$

to obtain p' arbitrarily close to $(q + 1/2)/(s + 1)$ and finally

$$\tau_s^{\mathrm{var}}(q) \ge \sup_{0<q'\le q} \min\left(1, q' + 1/2, \frac{q}{s + q'/\min(1, q' + 1/2)}\right) = \frac{q + 1/2}{s+1}.$$

In the case

$$q \ge s + 1/2$$

we choose

$$q' = 1/2$$

to obtain p' arbitrarily close to one and

$$\tau_s^{\mathrm{var}}(q) = 1.$$

The latter analysis and similar arguments for single-level algorithms, see Remark 5, imply the following facts, which correspond to upper bounds for the minimal errors. These upper bounds are achieved constructively, see Remark 3 and [8].

Corollary 2. *We have*

$$\min\left(\frac{q}{q+s}, \frac{q}{2q/(2q+1)+s}\right) \le \lambda_s^{\mathrm{fix}}(B(K))$$

and

$$\min\left(\frac{q}{s}, \frac{q+1/2}{s+1}, 1\right) \le \lambda_s^{\mathrm{var}}(B(K)).$$

It is interesting to compare Corollary 1 and Corollary 2 with the following upper and lower bounds for $\lambda_s^*(B(K))$, i.e., with lower and upper bounds for the minimal error $e_s^*(N, B(K))$, see [10, p. 18].

Theorem 5 (Kuo et al. (2009))**.** *We have*

$$\max\left(\frac{q}{q+1}, \min\left(\frac{q}{q+s}, \frac{q}{2q/(2q+1)+s}\right)\right) \le \lambda_s^*(B(K)) \le \min\left(\frac{q}{\min(s,1)}, 1\right)$$

Let us consider the particular case $s = 1$. Then we have

$$\lambda_1^{\text{fix}}(B(K)) = \frac{q}{q+1}$$

if $q \geq 1/2$, while we only know that

$$\frac{q}{2q/(2q+1)+1} \leq \lambda_1^{\text{fix}}(B(K)) \leq \frac{q}{q+1}$$

otherwise. Hence the single-level algorithm according to Remark 5 is optimal for fixed subspace sampling in the case $q \geq 1/2$ and close to being optimal for $q < 1/2$, since the fraction of the lower and upper bound for $\lambda_s^{\text{fix}}(B(K))$ is at least 0.93 for $q \in]0, 1/2[$. For variable subspace sampling

$$\lambda_1^{\text{var}}(B(K)) = \min(q, 1) \tag{22}$$

for $|q-1| \geq 1/2$, which shows that the multi-level algorithm according to Remark 3 is optimal in this case. For $|q-1| < 1/2$ we only know that

$$\frac{q+1/2}{2} \leq \lambda_1^{\text{var}}(B(K)) \leq \min(q, 1) \tag{23}$$

with a significant gap between the upper and the lower bound. Still variable subspace sampling is superior to fixed subspace sampling for all $q > 0$. Finally, by Theorem 5,

$$\frac{q}{q+1} \leq \lambda_1^*(B(K)) \leq \min(q, 1). \tag{24}$$

The corresponding upper bound $N^{-q/(q+1)}$ for the minimal error $e_1^*(N, B(K))$ is achieved constructively by so-called changing dimension algorithms introduced in [10]. The bound is always larger than the corresponding error bound for a suitable multi-level algorithm, and it is close to or even coincides with the upper error bound for a suitable single-level algorithm. Combining (22), (24), and $\lambda_1^{\text{var}}(B(K)) \leq \lambda_1^*(B(K))$ we obtain

$$\begin{aligned} \lambda_1^{\text{var}}(B(K)) = \lambda_1^*(B(K)) &= \min(q, 1), \quad \text{if } |q-1| \geq 1/2, \\ \frac{q+1/2}{2} \leq \lambda_1^{\text{var}}(B(K)) \leq \lambda_1^*(B(K)) &\leq \min(q, 1), \quad \text{if } |q-1| \geq 1/2. \end{aligned}$$

Thus we have optimality of the multi-level algorithm even in the cost model from [10] for $|q-1| \geq 1/2$, and in the case $|q-1| < 1/2$ we get an improved lower bound $\lambda_1^*(B(K)) \geq \frac{q+1/2}{2}$ in (24). See Figure 1 for an illustration.

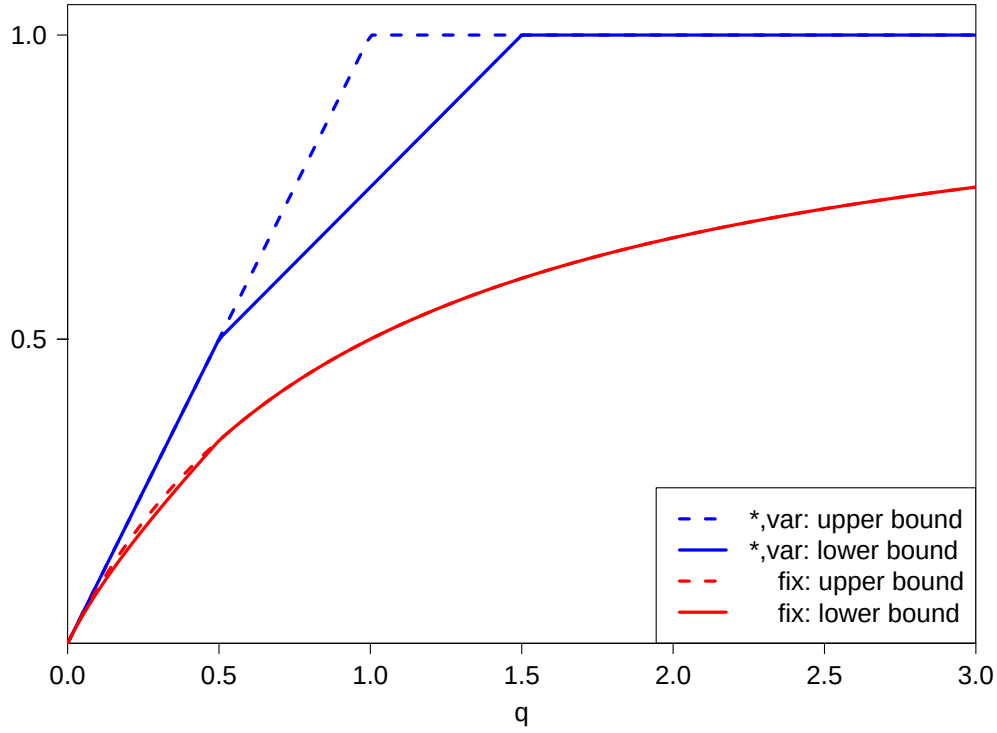

Figure 1: Upper and lower bounds for the exponents $\lambda_1^\dagger(B(K))$ in the case $K = \min$.

We stress, however, that for every fixed $q > 0$

$$\lim_{s\to\infty} \lambda_s^{\mathrm{var}}(B(K)) = 0,$$

while

$$\inf_{s>0} \lambda_s^*(B(K)) \geq \frac{q}{q+1} > 0.$$

Consequently, for large values of s, the changing dimension algorithm together with the cost model from [10] outperforms variable subspace sampling.

Niederreiter (T, d)-nets may serve as well as building blocks for the multilevel construction. In fact, if $q \geq q' > 1/2$ then the corresponding equal weight quadrature formulas yield any exponent

$$p' < \min(1, q'/2 + 1/4)$$

in the strong tractability estimate (16), see [17]. Exploiting this range of parameters in the optimization problem (20) we obtain

$$\tau_s^{\mathrm{var}}(q) \geq \min\left(\frac{q}{s+1}, \frac{q+1/2}{s+2}, 1\right).$$

Due to Corollary 2 this bound is suboptimal as long as $q < s + 3/2$. For larger values of q, however, Niederreiter (T, d)-nets together with the multi-level construction achieve optimality, see Corollary 1.

Remark 6. Although the above discussion focuses on ρ as the uniform distribution on $D = [0, 1]$, the results derived in the previous sections can also be applied to the case where ρ is the Gaussian distribution on $D = \mathbb{R}$, as suggested by the option pricing example. See [17, 7] for relevant kernels, k, and strong tractability results for this case.

Remark 7. The lattice and net designs discussed previously have the advantage of being extensible in both dimension and sample size. This allows one to use parts of one large design for each level of the multi-level algorithm. Specifically, one can remove the superscript (ℓ) labeling the design points in Remark 2 and re-arrange the terms to arrive at an equivalent formula that uses fewer arithmetic operations.

To illustrate this fact assume $n_1 > n_2 > \cdots > n_L > n_{L+1} = 0$, and put $d_0 = 0$, which implies $f(\mathbf{x}_{1:d_0}, \mathbf{c}) = 0$ for every $f \in H(K)$ and every $\mathbf{x} \in \mathcal{X}$. Then the multi-level algorithm Q satisfies

$$\begin{aligned}
Q(f) &= \sum_{\ell=1}^{L} \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \big(f((\mathbf{x}_i)_{1:d_\ell}, \mathbf{c}) - f((\mathbf{x}_i)_{1:d_{\ell-1}}, \mathbf{c})\big) \\
&= \sum_{\ell=1}^{L} \sum_{k=\ell}^{L} \sum_{i=n_{k+1}+1}^{n_k} \frac{1}{n_\ell} \big(f((\mathbf{x}_i)_{1:d_\ell}, \mathbf{c}) - f((\mathbf{x}_i)_{1:d_{\ell-1}}, \mathbf{c})\big) \\
&= \sum_{k=1}^{L} \sum_{i=n_{k+1}+1}^{n_k} \sum_{\ell=1}^{k} \frac{1}{n_\ell} \big(f((\mathbf{x}_i)_{1:d_\ell}, \mathbf{c}) - f((\mathbf{x}_i)_{1:d_{\ell-1}}, \mathbf{c})\big) \\
&= \sum_{k=1}^{L} \sum_{i=n_{k+1}+1}^{n_k} \left(\frac{1}{n_k} f((\mathbf{x}_i)_{1:d_k}, \mathbf{c}) + \sum_{\ell=1}^{k-1} \left(\frac{1}{n_\ell} - \frac{1}{n_{\ell+1}}\right) f((\mathbf{x}_i)_{1:d_\ell}, \mathbf{c})\right) \\
&= \sum_{k=1}^{L} \left(\frac{1}{n_k} \sum_{i=n_{k+1}+1}^{n_k} f((\mathbf{x}_i)_{1:d_k}, \mathbf{c}) + \sum_{\ell=1}^{k-1} \frac{n_{\ell+1} - n_\ell}{n_\ell \cdot n_{\ell+1}} \sum_{i=n_{k+1}+1}^{n_k} f((\mathbf{x}_i)_{1:d_\ell}, \mathbf{c})\right).
\end{aligned}$$

We observe that for each point $\mathbf{x}_i \in \mathcal{X}$ of the design with $n_{\ell+1} < i \leq n_\ell$ one only uses the first d_ℓ component. Moreover,

$$c_s^{\mathrm{var}}(Q) \leq \sum_{\ell=1}^{L} n_\ell \cdot d_\ell^s,$$

cf. (7). The savings here does not affect order of operations required for the multi-level algorithm but will have an affect on leading constants.

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