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„Extraktion quantifizierbarer Information aus komplexen Systemen"

An Analysis of Electrical Impedance Tomography with Applications to Tikhonov Regularization

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Abstract

This paper analyzes several regularization formulations arising in the continuum model /complete electrode model for the electrical impedance tomography inverse problem of determining the conductivity parameter from boundary measurements. The formulations incorporate a priori information of smoothness/sparsity on the inhomogeneity through Tikhonov regularization. The continuity and differentiability of the forward operator with respect to the conductivity parameter in L_p-norms is proved. These analytical results enable analyzing several popular formulations for both linearized and nonlinear models, and some important properties, e.g., existence, stability, consistency and convergence rates, are established. This provides some theoretical justifications of their practical usage.
Key words: electrical impedance tomography, Tikhonov regularization, convergence rate

1 Introduction

Electrical impedance tomography (EIT) is a diffusive imaging modality, and it has attracted considerable interest in noninvasive imaging and nondestructive testing. For instance, the reconstructions can be used for diagnostic purposes in medical applications, e.g., monitoring of lung function, detection of cancer in the skin and breast and location of epileptic foci [4]. A similar inverse problem arises in geophysics, where one uses electrodes on the surface of the earth or in bore holes to locate resistivity anomalies, e.g., minerals or contaminated sites, and it is known as geophysical resistivity tomography. Other applications include monitoring oil/gas mixture in oil pipelines and flow measurement in pneumatic conveying [51].

A typical experimental setup is as follows. One first applies an electrical current through the electrodes attached to the surface of the object, and then measures the resulting electrical potential on the boundary. In practice, the procedure is repeated several times with different input currents, which yields a partial information about the Neumann-to-Dirichlet map. EIT aims at determining a spatially-varying unknown physical electrical conductivity of the object by using these noisy measurements.

This inverse problem was first formulated by Calderón [9], who also gave a first uniqueness result for the linearized problem. The mathematical theory of unique solvability of the inverse problem with the complete Neumann-to-Dirichlet map has received considerable attention, and many profound theoretical results have been obtained. For a comprehensive overview of uniqueness results, we refer to the survey [54] and also [29]. The stability issues of the inverse problem have also been extensively investigated [1].

As typical of many inverse problems, EIT suffers from strong nonlinearity and severe ill-posedness. However, its broad prospective applications have aroused much interest in designing numerical techniques for its efficient solution. A large number of numerical methods have been proposed in the literature [56, 57, 11, 49, 2, 42, 48, 40, 13, 38]. Some of them are of variational type, i.e., based on minimizing a certain functional, typically the squared L_2 norm of the difference between the simulated boundary electrical

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potential due to an assumed conductivity and the measured potential. One standard approach of this type is described in [11], which applies one step of a Newton method with a constant conductivity as the initial guess, see also [38]. Due to the ill-posedness of the inverse problem, some sort of regularization is beneficial to combat the numerical instability inherent to the inverse problem. In [42], the standard Tikhonov regularization for EIT in unbounded domains was studied, and the consistency, stability and convergence rates were established. In [48], the Mumford-Shah model was suggested in the hope of simultaneously segmenting the conductivity image and enhancing the resolution. In [31], a level set approach was developed for estimating the interface, and in [13], it was used for total variation (TV) regularized L_2 data fitting to reconstruct a piecewise constant conductivity. Powerful analytical machineries, e.g., spectral analysis, can also lead to interesting reconstruction algorithms, e.g., factorization method [25, 37] and d-bar method [30]. These studies focus on deterministic inversion techniques. Alternatively, statistical inversion method [34] sheds interesting insights into reconstruction procedures, and has also gained some interest.

Amongst existing approaches using Tikhonov regularization, a penalty on the sought-for conductivity distribution is often incorporated. However, their mathematical properties and convergence behaviors of related algorithms have not received due attention despite their wide-spread practical adoption, and these procedures are largely applied in an ad hoc manner. As to relevant theoretical works, we are aware of [16, 48, 42, 47]. In [16], a regularization approach based on wavelet was analyzed, especially the convergence of an iterative algorithm was established. However, the analysis allows only continuously differentiable conductivities. In [42], the consistency, stability and convergence rates were discussed under a high Sobolev regularity on the conductivity, which can possibly make its numerical realization inconvenient. In the interesting and important works [48, 47], the existence and stability were established for Mumford-Shah and TV penalties, respectively, and also consistency [47]. Therefore, the issue of convergence rate remains largely unexplored, and the popular H^1-smoothness penalty has not been addressed.

Also there is still an interest in developing new reconstruction procedures that yield images of better resolution, especially by means of developing an increased focus on identifying useful information and on fully exploiting a priori knowledge. For conductivity fields that consist of an unknown but essentially uninteresting background plus a number of interesting features that have relatively "simple" mathematical descriptions, the ideas related to sparsity seem to offer a promising way forward that could, in principle, be adapted to develop a number of reconstruction techniques. Typically, the idea is incorporated by including a sparsity-promoting ℓ_1-penalty in the Tikhonov functional [14]. Recently, numerical experiments [32] have demonstrated its great potentials in that it is capable of reconstructing nonconvex/multiple inclusions with their magnitudes reasonably retrieved, and we refer to [22] for evaluation on real experimental data.

In this paper, we attempt to provide partial theoretical justifications of these formulations, i.e., smoothness/sparsity penalty for linearized and nonlinear models, especially convergence rates, along the line of [48, 47]. We first develop necessary analytical machineries, including continuity and Fréchet differentiability of the forward operator in L_p-norms. This is achieved with the help of Meyers' celebrated gradient estimate [43, 24, 21]. Then we capitalize on recent theoretical developments for nonsmooth regularization [8, 46, 50, 27, 5, 41, 23, 45, 7] to derive well-posedness and convergence rate results.

Finally, when completing the manuscript, we got to know the interesting work [18], which analyzes the standard Tikhonov regularization for diffuse optical tomography. Although the underlying ideas of [18] and the present work are similar, there are some significant differences in the forward model as well as the formulations: The optical tomography forward model in [18] has a Robin boundary condition and their focus is on the standard H^1-penalty, while the EIT model has a Neumann-type boundary condition and we are interested in smoothness/sparsity penalty.

The rest of the paper is organized as follows. We develop necessary analytical results, including continuity and differentiability of the forward operator with respect to L_p norms, for the continuum model in Section 2, which improve known results in the L_∞ norm, and enable us to apply regularization theory in a Hilbert space. In Section 3 the extension to the practically popular complete electrode model [52, 10, 6, 36] is discussed. Then in Section 4, we describe regularization formulations with ℓ_r penalties, and study their properties, e.g., existence, stability, consistency and convergence rate, under various conditions. The conventional smoothness penalty is covered as a special case. We conclude in Section 5 with a brief discussion on related issues.

2 Continuum model

This section studies the basic mathematical model, the continuum model, of the EIT problem. The main part is devoted to proving analytic properties of the parameter-to-state map and to establishing differentiability of the forward operator with respect to L_p norms.

2.1 Notation and definitions

Let Ω be an open bounded domain in $\mathbb{R}^d(d=2,3)$ with a Lipschitz boundary Γ. Throughout this paper, we shall make use of the space $\tilde{H}^1(\Omega)$, which is a subspace of the Sobolev space $H^1(\Omega)$ with vanishing mean on the boundary Γ, i.e., $\tilde{H}^1(\Omega)=\left\{v\in H^1(\Omega):\int_\Gamma v ds=0\right\}$. The spaces $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ are defined similarly. These spaces are equipped with the usual norms.

In the absence of interior current source and in the electrostatic state, Maxwell's system describing electromagnetic fields inside the object reduces to the following elliptic equation

$$-\nabla\cdot(\sigma\nabla u)=0 \quad \text{in } \Omega \tag{1}$$

with a Neumann boundary condition $\sigma\frac{\partial u}{\partial n}=j\in\tilde{H}^{-\frac{1}{2}}(\Gamma)$ on the boundary Γ. We normalize the solution by enforcing $\int_\Gamma u ds=0$ to ensure a unique solution $u\in\tilde{H}^1(\Omega)$, and denote by $F(\sigma)$ the forward operator.

In an EIT experiment, one measures a noisy version $\phi^\delta\in L_2(\Gamma)$ of the potential $\phi^\dagger=F(\sigma^\dagger)j$ ($\sigma^\dagger$ refers to the exact conductivity), with its accuracy measured by the noise level $\delta=\|\phi^\dagger-\phi^\delta\|_{L_2(\Gamma)}$. For a given pair (j,ϕ^δ), the variational approaches seek an approximation to $\sigma^\dagger$ by minimizing a certain discrepancy functional together with an appropriate penalty over the following admissible set

$$\mathcal{A}=\{\sigma\in L_\infty(\Omega):\lambda\le\sigma\le\lambda^{-1}\text{ a.e. and }\operatorname{supp}(\sigma-\sigma^\dagger)\subset\Omega'\},$$

for some fixed $\lambda\in(0,1)$, where Ω' is an open subset with a smooth boundary compactly contained in Ω. We shall endow the set $\mathcal{A}$ with an $L_p(\Omega)$ norm $(1\le p\le\infty)$.

Remark 2.1. *The admissible set $\mathcal{A}$ is closed and convex, but it has no interior points, i.e., for any $\sigma\in\mathcal{A}$, the ball $B_\epsilon=\{\tilde\sigma:\|\tilde\sigma-\sigma\|_{L_p(\Omega)}\le\epsilon\}$ is not completely contained in $\mathcal{A}$ for any $\epsilon>0$. Therefore, all the results presented below should be understood with respect to the relative topology.*

In what follows C denotes some generic constant, which may differ at different occurrences. We first state a well-known regularity estimate.

Lemma 2.1. *For any $\sigma\in\mathcal{A}$, the forward solution $F(\sigma)j$ satisfies*

$$\|F(\sigma)j\|_{H^1(\Omega)}\le C\|j\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. The estimate follows directly from Lax-Milgram theorem. □

Lemma 2.1 implies that the forward operator $F(\sigma)$ is uniformly bounded for a fixed j. We also recall the following norm equivalence result.

Lemma 2.2. *On the space $\tilde{H}^1(\Omega)$, the standard $H^1(\Omega)$ norm is equivalent to the $H^1(\Omega)$ semi-norm, i.e., there exist two constants c_0 and c_1 such that for any $v\in\tilde{H}^1(\Omega)$*

$$c_0\|v\|_{H^1(\Omega)}\le\|\nabla v\|_{L_2(\Omega)}\le c_1\|v\|_{H^1(\Omega)}.$$

For simplicity, we describe only results for one fixed input current j, and suppress the dependence of the solution $u=F(\sigma)j$ on j hereon. The extension to multiple data set, i.e., $\{(j_i,\phi_i)\}_{i=1}^N$, or the Neumann-to-Dirichlet map is straightforward. We remind that, in the latter case, the norm in the discrepancy should be understood as the operator norm, see e.g., [48, 47].

2.2 Continuity and differentiability

The differentiability of the forward operator $F(\sigma)$ with respect to the L_∞-norm is well-known. For instance, it was already noted in the pioneering work [9] that the forward map is analytic in σ with respect to the L_∞-norm. In [15], the Frechét differentiability was proved, and a general framework was provided in [39]. However, these results are concerned with L_∞ differentiability, which is insufficient for analyzing some Tikhonov functionals, including conventional H^1-smoothness/sparsity penalty.

We will derive a differentiability result in L_p norms, which is crucial for analyzing the formulations in Section 4. We shall prove continuity and Fréchet differentiability by applying Meyers' celebrated gradient estimates [43]. The proof techniques in this part are inspired by and closely follow [48], see also [15]. We start by stating Meyers' gradient estimate [43] as formulated in [48].

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d (d \geq 2)$. Assume that $\sigma \in L_\infty(\Omega)$ satisfies $\lambda < \sigma < \lambda^{-1}$ for some fixed $\lambda \in (0, 1)$. For $f \in (L_q(\Omega))^d$ and $h \in L_q(\Omega)$, let $u \in H^1(\Omega)$ be a weak solution of*

$$-\nabla \cdot (\sigma \nabla u) = -\nabla \cdot f + h \quad \text{in } \Omega.$$

Then, there exists a constant $Q \in (2, \infty)$ depending on λ and d only, $Q \to 2$ as $\lambda \to 0$ and $Q \to \infty$ as $\lambda \to 1$, such that for any $2 < q < Q$ we obtain $u \in W^{1,q}_{\text{loc}}(\Omega)$ and for any $\Omega_1 \subset\subset \Omega$

$$\|u\|_{W^{1,q}(\Omega_1)} \leq C(\|u\|_{H^1(\Omega)} + \|f\|_{L_q(\Omega)} + \|h\|_{L_q(\Omega)}),$$

where the constant C depends on λ, d, q, Ω_1 and Ω.

In this theorem the boundary condition for the differential equation can be general. Its effect enters the $W^{1,q}$-estimate through the term $\|u\|_{H^1(\Omega)}$. Otherwise, no further regularity has been assumed on the conductivity σ. In general, a precise estimate of the constant $Q(\lambda, d)$ is missing, although in the two-dimensional case, a sharp estimate of $Q(\lambda, d)$ was derived in [3]. We shall denote by $Q(\lambda)$ the number defined in Theorem 2.1 by suppressing its dependence on d.

Assisted with Theorem 2.1 and by repeatedly applying Hölder's inequality, we show the continuity of the operator $F(\sigma)$ with respect to the L_p norm for any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$.

Lemma 2.3. *For the operator $F(\sigma)$ and $\sigma, \sigma + \vartheta \in \mathcal{A}$, we have the following continuity estimate.*

(a) For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$ and $\sigma, \sigma + \vartheta \in \mathcal{A}$ we have

$$\|F(\sigma + \vartheta) - F(\sigma)\|_{H^1(\Omega)} \leq C\|\vartheta\|_{L_p(\Omega')};$$

(b) For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, there exists $q \in (2, Q(\lambda))$ such that

$$\|F(\sigma + \vartheta) - F(\sigma)\|_{W^{1,q}(\Omega')} \leq C\|\vartheta\|_{L_p(\Omega')};$$

(c) For $p \geq 1$ and any $q \in (2, Q(\lambda))$ we have the following estimates

$$\lim_{\|\vartheta\|_{L_p(\Omega')} \to 0} \|F(\sigma + \vartheta) - F(\sigma)\|_{W^{1,q}(\Omega')} = 0.$$

Proof. For $\sigma, \sigma + \vartheta \in \mathcal{A}$, the weak formulations of $F(\sigma)$ and $F(\sigma + \vartheta)$ gives

$$\int_\Omega \sigma \nabla F(\sigma) \cdot \nabla v dx = \int_\Omega (\sigma + \vartheta) \nabla F(\sigma + \vartheta) \cdot \nabla v dx, \quad \forall v \in \tilde{H}^1(\Omega),$$

i.e.,

$$\int_\Omega \sigma \nabla (F(\sigma) - F(\sigma + \vartheta)) \cdot \nabla v dx = \int_\Omega \vartheta \nabla F(\sigma + \delta\sigma) \cdot \nabla v dx, \quad \forall v \in \tilde{H}^1(\Omega).$$

Taking $v = F(\sigma) - F(\sigma+\vartheta) \in \tilde{H}^1(\Omega)$ in the equation and noting the support of ϑ in Ω' gives

$$
\begin{aligned}
\int_\Omega \sigma |\nabla (F(\sigma) - F(\sigma+\vartheta))|^2 dx =& \int_\Omega \vartheta \nabla F(\sigma+\vartheta)\cdot\nabla(F(\sigma)-F(\sigma+\vartheta))dx\\
=& \int_{\Omega'} \vartheta \nabla F(\sigma+\vartheta)\cdot\nabla(F(\sigma)-F(\sigma+\vartheta))dx\\
\leq& \|\vartheta\|_{L_p(\Omega')}\|\nabla F(\sigma+\vartheta)\|_{L_q(\Omega')}\|\nabla(F(\sigma)-F(\sigma+\vartheta))\|_{L_2(\Omega)},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. The assumption $p\in(\frac{2Q(\lambda)}{Q(\lambda)-2},\infty]$ implies $q<Q(\lambda)$. By Theorem 2.1, there exists a constant C such that

$$
\|\nabla F(\sigma+\vartheta)\|_{L_q(\Omega')} \leq C\|F(\sigma+\vartheta)\|_{H^1(\Omega)} \leq C\|j\|_{H^{-\frac{1}{2}}(\Gamma)}.
$$

This together with Lemma 2.2 shows

$$
\|F(\sigma)-F(\sigma+\vartheta)\|_{H^1(\Omega)} \leq C\|\vartheta\|_{L_p(\Omega')}.
$$

This shows the first part of the lemma.

To prove the second part, we fix $q\in(2,Q(\lambda))$ and choose $p=\frac{2q}{q-2}$, i.e., $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $p\in(\frac{4Q(\lambda)}{Q(\lambda)-2},\infty]$. We apply Meyers' theorem and obtain

$$
\|F(\sigma)-F(\sigma+\vartheta)\|_{W^{1,q}(\Omega')} \leq C\left(\|F(\sigma)-F(\sigma+\vartheta)\|_{H^1(\Omega)} + \|\vartheta\nabla F(\sigma+\vartheta)\|_{L_q(\Omega)}\right). \tag{2}
$$

The first term in the bracket has been estimated in part (a), and thus we only need to bound the term $\|\vartheta\nabla F(\sigma+\vartheta)\|_{L_q(\Omega)}$. Take any small ϵ such that $q'=q+\epsilon\in(q,Q(\lambda))$ and $\frac{qq'}{q'-q}\geq p$. By appealing to Hölder's inequality, we deduce

$$
\begin{aligned}
\int_\Omega |\vartheta\nabla F(\sigma+\vartheta)|^q dx =& \int_{\Omega'} |\vartheta|^q|\nabla F(\sigma+\vartheta)|^q dx\\
\leq& \left(\int_{\Omega'}|\vartheta|^{\frac{qq'}{q'-q}}dx\right)^{1-\frac{q}{q'}}\left(\int_{\Omega'}|\nabla F(\sigma+\vartheta)|^{q'}dx\right)^{\frac{q}{q'}}\\
\leq& C\|j\|^q_{H^{-\frac{1}{2}}(\Omega)}\left(\int_{\Omega'}|\vartheta|^{\frac{qq'}{q'-q}}dx\right)^{1-\frac{q}{q'}},
\end{aligned} \tag{3}
$$

where we have applied Meyers' theorem to the term $\|\nabla F(\sigma+\vartheta)\|_{L_{q'}(\Omega')}$. The choice $\frac{qq'}{q'-q}\geq p$ implies

$$
\int_{\Omega'}|\vartheta|^{\frac{qq'}{q'-q}}dx = \int_{\Omega'}|\vartheta|^p\cdot|\vartheta|^{\frac{qq'}{q'-q}-p}dx \leq |\lambda|^{p-\frac{qq'}{q'-q}}\int_{\Omega'}|\vartheta|^p dx, \tag{4}
$$

Collecting the exponents in inequalities (2)-(4) yields

$$
\|F(\sigma+\vartheta)-F(\sigma)\|_{W^{1,q}(\Omega')} \leq C\|\vartheta\|^{\frac{q'-q}{q'q}p}_{L_p(\Omega')}.
$$

The choice of $p\in(\frac{4Q(\lambda)}{Q(\lambda)-2},\infty]$ indicates that $q\in(2,\frac{4Q(\lambda)}{2+Q(\lambda)})\subset(2,Q(\lambda))$, and thus $q<q'=\frac{2q}{4-q}<Q(\lambda)$. With this choice of q', we have $\frac{q'-q}{q'q}p=1$, i.e.,

$$
\|F(\sigma+\vartheta)-F(\sigma)\|_{W^{1,q}(\Omega')} \leq C\|\vartheta\|_{L_p(\Omega')},
$$

which shows the second assertion.

Now we turn to the third assertion. By the proof of the second assertion, for any $q\in(2,Q(\lambda))$ and $q'\in(q,Q(\lambda))$ we have

$$
\|\vartheta\nabla F(\sigma+\vartheta)\|_{L_q(\Omega')} \leq C\|j\|_{H^{-\frac{1}{2}}(\Gamma)}\|\vartheta\|_{L_{\frac{qq'}{q'-q}}(\Omega')}.
$$

If the exponent $\tilde{p} = \frac{qq'}{q'-q} \leq p$, then by Hölder's inequality, we have

$$\|\vartheta\|_{L_{\tilde{p}}(\Omega')} \leq |\Omega'|^{\frac{p-\tilde{p}}{p\tilde{p}}} \|\vartheta\|_{L_p(\Omega')}.$$

As to the other case, by the $L_\infty(\Omega)$ boundedness of the admissible set $\mathcal{A}$ we deduce for $\tilde{p} \geq p \geq 1$

$$\|\vartheta\|_{L_{\tilde{p}}(\Omega')}^{\tilde{p}} \leq C\|\vartheta\|_{L_p(\Omega')}^{p}.$$

In either case, we have that the term goes to zero as $\|\vartheta\|_{L_p(\Omega')} \to 0$. Hence, we obtain

$$\lim_{\|\vartheta\|_{L_p(\Omega')}\to 0} \|F(\sigma+\vartheta) - F(\sigma)\|_{W^{1,q}(\Omega')} = 0$$

for arbitrary $p \geq 1$. $\square$

Remark 2.2. *For the case of $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \frac{4Q(\lambda)}{Q(\lambda)-2})$ and $\frac{1}{q} = \frac{1}{2} - \frac{1}{p}$, we have by choosing $q' = Q(\lambda) - \epsilon$*

$$\|F(\sigma+\vartheta) - F(\sigma)\|_{W^{1,q}(\Omega')} \leq C\|\vartheta\|_{L_p(\Omega')}^{r} \quad \textit{with} \quad r = \frac{2(Q(\lambda)-\epsilon-q)}{(Q(\lambda)-\epsilon)(q-2)}.$$

It is easy to see that $r < \frac{2Q(\lambda)-q}{Q(\lambda)(q-2)} \leq 1$.

Let us now proceed to the differentiability of the forward operator. We fix $\sigma \in \mathcal{A}$, and let ϑ be a perturbation to σ belonging to $L_\infty(\Omega')$ and extended by zero outside Ω'. Let $w \in \tilde{H}^1(\Omega)$ be the weak solution to

$$\int_\Omega \sigma\nabla w \cdot \nabla v dx = -\int_\Omega \vartheta\nabla F(\sigma) \cdot \nabla v dx \quad \forall v \in \tilde{H}^1(\Omega).$$

The above equation is the linearized problem of the Neumann forward problem at σ. We shall call $F'(\sigma) : L_p(\Omega') \to \tilde{H}^1(\Omega)$ the map from ϑ to w.

Lemma 2.4. *For any $\sigma \in \mathcal{A}$, the linear mapping $F'(\sigma)$ defined above has the following continuity properties*

- *(a) For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$, the operator $F'(\sigma) : L_p(\Omega') \to \tilde{H}^1(\Omega)$ is bounded;*
- *(b) For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, there exists a $q \in (2, Q(\lambda))$ such that $F'(\sigma) : L_p(\Omega') \to W^{1,q}(\Omega')$ is bounded;*
- *(c) For $p \geq 1$ and any $q \in (2, Q(\lambda))$*

$$\lim_{\|\vartheta\|_{L_p(\Omega')}\to 0} \|F'(\sigma)\vartheta\|_{W^{1,q}(\Omega')} = 0.$$

Proof. For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$, we can choose q by $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$, i.e., $q \in (2, Q(\lambda))$. By the weak formulation of $F'(\sigma)\vartheta$ and the generalized Hölder inequality, we have

$$\begin{aligned}
\int_\Omega \sigma|\nabla F'(\sigma)\vartheta|^2 dx &= -\int_\Omega \vartheta\nabla F(\sigma) \cdot \nabla F'(\sigma)\vartheta dx \\
&= -\int_{\Omega'} \vartheta\nabla F(\sigma) \cdot \nabla F'(\sigma)\theta dx \\
&\leq \|\vartheta\|_{L_p(\Omega')}\|\nabla F(\sigma)\|_{L_q(\Omega')}\|\nabla F'(\sigma)\vartheta\|_{L_2(\Omega)} \\
&\leq C\|\vartheta\|_{L_p(\Omega')}\|F(\sigma)\|_{H^1(\Omega)}\|\nabla F'(\sigma)\vartheta\|_{L_2(\Omega)}.
\end{aligned}$$

This together with Lemma 2.2 implies that the operator $F'(\sigma):L_p(\Omega')\to\tilde{H}^1(\Omega)$ is bounded, thereby showing the first assertion. To prove the second and third assertions, we appeal to Meyers' theorem to derive

$$\|F'(\sigma)\vartheta\|_{W^{1,q}(\Omega')}\le C\left(\|F'(\sigma)\vartheta\|_{H^1(\Omega)}+\|\vartheta\nabla F(\sigma)\|_{L_q(\Omega)}\right).$$

Therefore, we need to bound the term $\|\vartheta\nabla F(\sigma)\|_{L_q(\Omega)}$, which can be estimated as in Lemma 2.3. This shows assertions (b) and (c). ☐

The next result shows differentiability of the operator F.

Theorem 2.2. *Let $p\in(\frac{2Q(\lambda)}{Q(\lambda)-2},\infty]$. Then the forward operator F is differentiable in the sense that for any $\sigma,\sigma+\vartheta\in\mathcal{A}$ there holds*

$$\frac{\|F(\sigma+\vartheta)-F(\sigma)-F'(\sigma)\vartheta\|_{H^1(\Omega)}}{\|\vartheta\|_{L_p(\Omega')}}\to 0 \text{ as } \vartheta\to 0 \text{ in } L_p(\Omega').$$

Proof. Since ϑ vanishes on the boundary, the function $w=F(\sigma+\vartheta)-F(\sigma)-F'(\sigma)\vartheta\in\tilde{H}^1(\Omega)$ satisfies

$$\int_\Omega(\sigma+\vartheta)\nabla w\cdot\nabla v dx=-\int_\Omega\vartheta\nabla F'(\sigma)\vartheta\cdot\nabla v dx\quad\forall\, v\in\tilde{H}^1(\Omega).$$

Taking $v=w$ in the weak formulation gives

$$\begin{aligned}\int_\Omega(\sigma+\vartheta)|\nabla w|^2dx&=-\int_\Omega\vartheta\nabla F'(\sigma)\vartheta\cdot\nabla w dx=-\int_{\Omega'}\vartheta\nabla F'(\sigma)\vartheta\cdot\nabla w dx\\&\le\|\vartheta\|_{L_p(\Omega')}\|\nabla F'(\sigma)\vartheta\|_{L_q(\Omega')}\|\nabla w\|_{L_2(\Omega')}.\end{aligned}$$

We observe that $p\in(\frac{2Q(\lambda)}{Q(\lambda)-2},\infty]$ implies $2\le q<Q(\lambda)$. By applying Lemma 2.4(c) to the term $\|\nabla F'(\sigma)\vartheta\|_{L_q(\Omega')}$, we arrive at the desired assertion. ☐

The Lipschitz continuity of the operator $F'(\sigma)$ is essential, e.g., in studying convergence rates of regularization methods [19] and in analyzing iterative algorithms, and it will be used in Section 4.

Theorem 2.3. *For any $p\in(\frac{4Q(\lambda)}{Q(\lambda)-2},\infty]$, the operator $F'(\sigma)$ is Lipschitz continuous with respect to $L_p(\Omega')$ in the sense that for any $\sigma,\sigma+\vartheta\in\mathcal{A}$*

$$\|F'(\sigma+\vartheta)-F'(\sigma)\|_{L(L_p(\Omega'),\tilde{H}^1(\Omega))}\le C\|\vartheta\|_{L_p(\Omega')}.$$

Proof. For any $\varsigma\in L_p(\Omega')$, by the weak formulations for $F'(\sigma)\varsigma$ and $F'(\sigma+\vartheta)\varsigma$, i.e.,

$$\begin{aligned}\int_\Omega\sigma\nabla F'(\sigma)\varsigma\cdot\nabla v dx&=-\int_\Omega\varsigma\nabla F(\sigma)\cdot\nabla v dx\quad\forall v\in\tilde{H}^1(\Omega),\\\int_\Omega(\sigma+\vartheta)\nabla F'(\sigma+\vartheta)\varsigma\cdot\nabla v dx&=-\int_\Omega\varsigma\nabla F(\sigma+\vartheta)\cdot\nabla v dx\quad\forall v\in\tilde{H}^1(\Omega),\end{aligned}$$

we derive that $w=F'(\sigma+\vartheta)\varsigma-F'(\sigma)\varsigma\in\tilde{H}^1(\Omega)$ satisfies

$$\int_\Omega\sigma\nabla w\cdot\nabla v dx=-\int_\Omega\varsigma\nabla(F(\sigma+\vartheta)-F(\sigma))\cdot\nabla v dx-\int_\Omega\vartheta\nabla F'(\sigma+\vartheta)\varsigma\cdot\nabla v dx\quad\forall v\in\tilde{H}^1(\Omega).$$

Letting $v=w$ and applying the generalized Hölder's inequality with q satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $q\in(2,Q(\lambda))$ to the two terms on the right hand side of the above identity, we get

$$\begin{aligned}-\int_\Omega\varsigma\nabla(F(\sigma+\vartheta)-F(\sigma))\cdot\nabla w dx&\le\|\varsigma\|_{L_p(\Omega')}\|\nabla(F(\sigma+\vartheta)-F(\sigma))\|_{L_q(\Omega')}\|\nabla w\|_{L_2(\Omega')}\\&\le C\|\varsigma\|_{L_p(\Omega')}\|\vartheta\|_{L_p(\Omega')}\|\nabla w\|_{L_2(\Omega')},\end{aligned}$$

and

$$-\int_\Omega \vartheta\nabla F'(\sigma+\vartheta)\varsigma\cdot\nabla w dx \leq \|\vartheta\|_{L_p(\Omega')}\|\nabla F'(\sigma+\vartheta)\varsigma\|_{L_q(\Omega')}\|\nabla w\|_{L_2(\Omega')}$$
$$\leq C\|\varsigma\|_{L_p(\Omega)}\|\vartheta\|_{L_p(\Omega')}\|\nabla w\|_{L_2(\Omega')},$$

in view of Lemmas 2.3(b) and 2.4(b). Combining these two estimates gives

$$\|\nabla w\|_{L_2(\Omega)} \leq C\|\varsigma\|_{L_p(\Omega')}\|\vartheta\|_{L_p(\Omega')},$$

which shows the desired assertion. □

Remark 2.3. *For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, by Theorem 2.3 and trace theorem [20], we have the following estimate for the linear approximation of the operator $F(\sigma)$*

$$\|F(\sigma+\vartheta)-F(\sigma)-F'(\sigma)\vartheta\|_{L_2(\Gamma)} \leq \tfrac{L}{2}\|\vartheta\|^2_{L_p(\Omega')},$$

where L is the Lipschitz constant of the operator $F'(\sigma)$, which depends on the constant from Meyers' estimate and the Sobolev embedding constant.

The adjoint of the operator $F'(\sigma)$ is very useful in analyzing the convergence rate as well as in deriving the gradient of the discrepancy functional. We have the following representation, where p^* is the conjugate exponent of $p \geq 1$, i.e., $\frac{1}{p}+\frac{1}{p^*}=1$.

Theorem 2.4. *The adjoint of the operator $F'(\sigma): L_p(\Omega') \to L_2(\Gamma)$ is given by*

$$(F'(\sigma))^*: L_2(\Gamma) \to L_{p^*}(\Omega'),$$
$$f \mapsto -\nabla\tilde{u}\cdot\nabla F(\sigma),$$

where $\tilde{u} \in \tilde{H}^1(\Omega)$ solves the adjoint problem

$$\int_\Omega \sigma\nabla\tilde{u}\cdot\nabla v dx = \int_\Gamma f v ds \quad \forall v \in \tilde{H}^1(\Omega).$$

Proof. For any $\vartheta \in L_p(\Omega')$, letting $v = \tilde{u}$ and $v = F'(\sigma)\vartheta$ in the weak formulations for $F'(\sigma)\vartheta$ and $\tilde{u}$ respectively gives

$$\int_{\Omega'} -\vartheta\nabla F(\sigma)\cdot\nabla\tilde{u} dx = \int_\Gamma F'(\sigma)\vartheta f ds,$$

which shows the desired assertion. □

In summary, we have the following useful corollary.

Corollary 2.1. *If $d=2$, or if $d=3$ and additionally λ is sufficiently close to one, then the operator $F(\sigma)$ is differentiable, and the operator $F'(\sigma)$ is Lipschitz continuous with respect to the topology of $H^1(\Omega')$.*

Proof. By the Sobolev embedding theorem [20], we have

$$H^1(\Omega') \hookrightarrow L_s(\Omega') \text{ for any } \begin{cases} s<\infty, & d=2, \\ s\leq 6, & d=3. \end{cases}$$

Therefore the result holds naturally for $d=2$. In case of $d=3$, we need some $Q(\lambda)>2$ such that $\frac{4Q(\lambda)}{Q(\lambda)-2}<6$, i.e. $Q(\lambda)>6$, according to Lemma 2.3 and Theorem 2.3. By Meyers' theorem, we have $Q(\lambda)\to\infty$ as $\lambda\to 1$ and $Q(\lambda)$ depends continuously on λ [43]. Therefore for λ sufficiently close to 1, we have $Q(\lambda)>6$ as desired. □

Remark 2.4. *Note that the classical L_∞ estimates do not imply Corollary 2.1 since $H^1(\Omega')$ does not embed continuously into $L_\infty(\Omega')$ for $d=2,3$, and thus the L_p estimates derived here are advantageous for justifying regularization in the Hilbert space $H^1(\Omega')$.*

The continuity and differentiability results can be used to study various discrepancy functionals. The following result shows the continuity of the standard least-squares discrepancy $J(\sigma) = \frac{1}{2}\|F(\sigma) - \phi^\delta\|^2_{L_2(\Gamma)}$.

Proposition 2.1. *The functional J is Hölder continuous with respect to $L_p(\Omega')$ for any $1 \le p \le \infty$.*

Proof. First we fixed $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$. For any $\sigma, \sigma + \vartheta \in \mathcal{A}$, by Cauchy-Schwarz inequality, we observe

$$|J(\sigma+\vartheta) - J(\sigma)| \le \tfrac{1}{2}\|F(\sigma) - F(\sigma+\vartheta)\|_{L_2(\Gamma)}\|F(\sigma) + F(\sigma+\vartheta) - 2\phi^\delta\|_{L_2(\Gamma)}.$$

The choice $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$ implies the existence of a $q \in (2, Q(\lambda))$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. By trace theorem [20], Lemma 2.3 and the generalized Hölder inequality, we can estimate the term $\|F(\sigma+\vartheta) - F(\sigma)\|_{L_2(\Gamma)}$ as

$$\|F(\sigma) - F(\sigma+\vartheta)\|_{L_2(\Gamma)} \le C\|F(\sigma+\vartheta) - F(\sigma)\|_{H^1(\Omega)} \le C\|\vartheta\|_{L_p(\Omega')}.$$

The term $\|F(\sigma)+F(\sigma+\vartheta)-2\phi^\delta\|_{L_2(\Gamma)}$ is uniformly bounded by Lemma 2.1 and trace theorem. Therefore, J is Lipschitz continuous with respect to $L_p(\Omega')$ for any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$. The Hölder continuity with respect to $L_p(\Omega')$ for any $1 \le p \le \infty$ follows from the $L_\infty(\Omega)$ boundedness of the set $\mathcal{A}$. □

Remark 2.5. *All the results presented in this section are for the admissible set $\mathcal{A}$, which allows only variation of the conductivity in the interior part Ω' of the domain Ω. This restriction can be removed by imposing higher regularity on the flux j, i.e., $j \in L_s(\Gamma) \cap \tilde{H}^{-\frac{1}{2}}(\Gamma)$ for sufficiently large s, and on the boundary Γ, see Theorem 3.1 in the next section. All the results remain valid with this modification.*

3 Complete electrode model

This section discusses relevant analytical results for the complete electrode model (CEM), presently the most accurate model. This model can achieve an accuracy comparable with experimental precision [12, 52, 10], and thus it is standard model for medical applications.

In contrast to the continuum model discussed earlier, the CEM utilizes nonstandard boundary conditions to capture important features of EIT experiments, e.g., discrete nature of electrodes, contact impedance effect and shunting effect. Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be an open domain with a smooth boundary, and $\{e_l\}_{l=1}^L \subset \Gamma$ be L electrodes, each with a positive surface measure. We assume that each electrode is connected and they are disjointed from each other, i.e., $\bar{e}_i \cap \bar{e}_j = \emptyset$ for $i \ne j$. Let $\mathbb{R}^L_\diamond := \{I \in \mathbb{R}^L : \sum_{l=1}^L I_l = 0\}$, and $H = H^1(\Omega) \oplus \mathbb{R}^L_\diamond$ with its norm defined by $\|(v, V)\|^2_H = \|v\|^2_{H^1(\Omega)} + \|V\|^2_{\mathbb{R}^L}$, which is equivalent to the norm defined by $\|(v, V)\|^2_* = \|\nabla v\|^2_{L_2(\Omega)} + \sum_{l=1}^L \|v - V_l\|^2_{L_2(\Gamma)}$ [52]. Then the model reads as follows: Given a current $I \in \mathbb{R}^L_\diamond$ and positive contact impedances $\{z_l\}_{l=1}^L$, find $(u, U) \in H$ such that

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega, \\ u + z_l \sigma \frac{\partial u}{\partial n} = U_l & \text{on } e_l,\ l = 1, 2, \ldots, L, \\ \int_{e_l} \sigma \frac{\partial u}{\partial n} ds = I_l & \text{for } l = 1, 2, \ldots, L, \\ \sigma \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \backslash \cup_{l=1}^L e_L. \end{cases} \tag{5}$$

Then the inverse problem consists of estimating the conductivity σ from the measured vector $U \in \mathbb{R}^L_\diamond$.

We first discuss the forward problem (5). The weak formulation is given by [52]: find $(u, U) \in H$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla v dx + \sum_{l=1}^L z_l^{-1} \int_{e_l} (u - U_l)(v - V_l) ds = \sum_{l=1}^L I_l V_l \quad \forall (v, V) \in H. \tag{6}$$

Lax-Milgram theorem yields the existence and uniqueness of a solution (u, U) [52]. We denote the solution operator by $F(\sigma)$, i.e., $(u, U) = (F_1(\sigma)I, F_2(\sigma)I) = F(\sigma)I \in H$. Again, we suppress the dependence on the input current I. The admissible set $\mathcal{A}$ for the conductivity σ is given by $\mathcal{A} = \{\sigma \in L_\infty(\Omega) : \lambda \leq \sigma \leq \lambda^{-1} \text{ a.e. } \Omega\}$ for some fixed constant $\lambda \in (0,1)$.

We first recall the following elliptic regularity estimate [21, Thm 2] [24], which is analogue of Theorem 2.1 for Neumann problems. It enables exploiting the higher regularity of the Neumann boundary condition in system (5) and thus establishing better regularity for the model.

Theorem 3.1. *For any $\sigma \in \mathcal{A}$, there exists a constant Q, which depends on d and λ only and tends to ∞ and 2 as $\lambda \to 1$ and $\lambda \to 0$, respectively, such that for any $q \in (2, Q)$, any $s \in [q - \frac{q}{d}, \infty]$ and $j \in L_s(\Gamma) \cap \tilde{H}^{-\frac{1}{2}}(\Gamma)$, $f \in (L_q(\Omega))^d$, the solution u to the Neumann problem*

$$-\nabla \cdot (\sigma \nabla u) = \nabla \cdot f \quad \text{in } \Omega \quad \text{and} \quad \sigma \frac{\partial u}{\partial n} = j \quad \text{on } \Gamma$$

satisfies the estimate

$$\|u\|_{W^{1,q}(\Omega)} \leq C \left(\|j\|_{L_s(\Gamma)} + \|f\|_{L_q(\Omega)} \right),$$

where C is a constant depending on d, λ, Ω and q only.

A first estimate is the uniform boundedness of the operator $F(\sigma)$.

Lemma 3.1. *The operator $F(\sigma) : \mathcal{A} \to H$ is uniformly bounded.*

Proof. Setting $(v, V) = (u, U) \in H$ in the weak formulation (6) gives

$$\begin{aligned}\lambda \|\nabla u\|^2_{L_2(\Omega)} + c_0 \sum_{l=1}^{L} \|u - U_l\|^2_{L_2(e_l)} &\leq \int_\Omega \sigma |\nabla u|^2 dx + \sum_{l=1}^{L} z_l^{-1} \int_{e_l} (u - U_l)^2 ds \\ &= \sum_{l=1}^{L} I_l U_l \leq \|I\|_{\mathbb{R}^L} \|(u, U)\|_H,\end{aligned}$$

where $c_0 = \min\{z_l^{-1}, l = 1, \ldots, L\}$. This shows the uniform boundedness of the operator $F(\sigma)$. □

The following theorem provides the key regularity result.

Theorem 3.2. *For any $\sigma \in \mathcal{A}$, there exists a constant $Q > 2$, depending on d and λ only and tending to ∞ and 2 as $\lambda \to 1$ and $\lambda \to 0$, respectively, such that the solution $(u, U) \in H$ to (5) satisfies for any $q \in (2, Q(\lambda))$*

$$\|u\|_{W^{1,q}(\Omega)} \leq C \|I\|_{\mathbb{R}^L},$$

where C is a constant depending on d, λ, Ω and q only.

Proof. By Lemma 3.1, there exists a solution $(u, U) \in H$ such that

$$\|(u, U)\|_H \leq C \|I\|_{\mathbb{R}^L}.$$

Next we rewrite equation (5) as

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega, \\ \sigma \frac{\partial u}{\partial n} = g, & \text{on } \Gamma, \end{cases}$$

where $g = \sum_{l=1}^{L} \frac{1}{z_l} (U_l - u) \chi_{e_l} \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. Note that $u \in H^1(\Omega)$ and U_l is a constant. By the Sobolev embedding theorem [20], we have $g \in L_s(\Gamma)$, $\forall s < \infty$ if $d = 2$ and $g \in L_4(\Gamma)$ if $d = 3$. In the case of $d = 2$, by Theorem 3.1, we have for any $q < Q(\lambda)$

$$\|\nabla u\|_{L_q(\Omega)} \leq C \|g\|_{L_s(\Gamma)} \leq C \|I\|_{\mathbb{R}^L},$$

by Lemma 3.1. In the case of $d = 3$, similarly by Theorem 3.1, we have for any $q < \min(Q(\lambda), 6)$ again $\|\nabla u\|_{L_q(\Omega)} \leq C \|I\|_{\mathbb{R}^L}$ holds. The proof of the theorem is concluded if $Q(\lambda) < 6$, otherwise we can repeat the procedure with the estimate $\|\nabla u\|_{L_q(\Omega)} \leq C \|I\|_{\mathbb{R}^L}$ for $q < 6$ and Sobolev embedding theorem [20] that $W^{1,r}(\Omega)$ with $r > 3$ embeds continuously into $L_\infty(\Gamma)$, and the theorem follows. □

With Theorem 3.2 at hand, we can state analogous continuity and differentiability results for the CEM forward operator $F(\sigma)$. Their proofs are identical with those in Section 2, and thus omitted. A first result is the following continuity result.

Lemma 3.2. *For the operator $F(\sigma)$ and $\sigma, \sigma + \vartheta \in \mathcal{A}$, we have the following continuity estimate.*

(a) *For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$ and $\sigma, \sigma + \vartheta \in \mathcal{A}$ we have*

$$\|F(\sigma + \vartheta) - F(\sigma)\|_H \leq C\|\vartheta\|_{L_p(\Omega)};$$

(b) *For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, there exists a $q \in (2, Q(\lambda))$ such that*

$$\|F_1(\sigma + \vartheta) - F_1(\sigma)\|_{W^{1,q}(\Omega)} \leq C\|\vartheta\|_{L_p(\Omega)};$$

(c) *For $p \geq 1$ and any $q \in (2, Q(\lambda))$ we have the following estimates*

$$\lim_{\|\vartheta\|_{L_p(\Omega)} \to 0} \|F_1(\sigma + \vartheta) - F_1(\sigma)\|_{W^{1,q}(\Omega)} = 0.$$

Let us now proceed to differentiability. As before, we fix $\sigma \in \mathcal{A}$, and let ϑ be a perturbation to σ belonging to $L_\infty(\Omega)$. Let $(w, W) \in H$ be the weak solution to

$$\int_\Omega \sigma \nabla w \cdot \nabla v dx + \sum_{l=1}^{L} z_l^{-1} \int_{e_l} (w - W_l)(v - V_l) ds = -\int_\Omega \vartheta \nabla F_1(\sigma) \cdot \nabla v dx \quad \forall (v, V) \in H.$$

The above equation is the linearized problem of the CEM forward problem at σ. We shall call $F'(\sigma) : L_p(\Omega) \to H$ the map from ϑ to (w, W). Then we have the following.

Lemma 3.3. *For any $\sigma \in \mathcal{A}$, the linear map $F'(\sigma)$ defined above has the following continuity properties.*

(a) *For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$, the operator $F'(\sigma) : L_p(\Omega) \to H$ is bounded;*

(b) *For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, there exists $q \in (2, Q(\lambda))$ such that $F_1'(\sigma) : L_p(\Omega) \to W^{1,q}(\Omega)$ is bounded;*

(c) *For $p \geq 1$ and any $q \in (2, Q(\lambda))$*

$$\lim_{\|\vartheta\|_{L_p(\Omega)} \to 0} \|F_1'(\sigma)\vartheta\|_{W^{1,q}(\Omega)} = 0.$$

Now we can state differentiability of the operator $F(\sigma)$, and Lipschitz continuity of the operator $F'(\sigma)$.

Theorem 3.3. *For any $\sigma, \sigma + \vartheta \in \mathcal{A}$, there hold*

(a) *For any $p \in (\frac{2Q(\lambda)}{Q(\lambda)-2}, \infty]$, the operator $F(\sigma)$ is differentiable in the sense*

$$\frac{\|F(\sigma + \vartheta) - F(\sigma) - F'(\sigma)\vartheta\|_H}{\|\vartheta\|_{L_p(\Omega)}} \to 0 \text{ as } \vartheta \to 0 \text{ in } L_p(\Omega);$$

(b) *For any $p \in (\frac{4Q(\lambda)}{Q(\lambda)-2}, \infty]$, the operator $F'(\sigma)$ is Lipschitz continuous in the sense*

$$\|F'(\sigma + \vartheta) - F'(\sigma)\|_{L(L_p(\Omega), H)} \leq C\|\vartheta\|_{L_p(\Omega)}.$$

As to the adjoint of the operator $F'(\sigma) : L_p(\Omega) \to \mathbb{R}^L_\diamond$, we have the following representation.

Theorem 3.4. *The adjoint of the operator $F'(\sigma) : L_p(\Omega) \to \mathbb{R}^L_\diamond$ is given by*

$$\begin{aligned}(F'(\sigma))^* : \mathbb{R}^L_\diamond &\to L_{p^*}(\Omega),\\ \tilde{I} &\mapsto -\nabla\tilde{u}\cdot\nabla F_1(\sigma),\end{aligned}$$

where $(\tilde{u}, \tilde{U}) \in H$ solves the adjoint problem

$$\int_\Omega \sigma\nabla\tilde{u}\cdot\nabla v dx + \sum_{l=1}^L z_l^{-1}\int_{e_l}(\tilde{u}-\tilde{U}_l)(v-V_l)ds = \sum_{l=1}^L \tilde{I}_l V_l \quad \forall(v,V)\in H.$$

In summary, we have the following corollary for the CEM forward operator $F(\sigma)$.

Corollary 3.1. *If $d = 2$, or if $d = 3$ and additionally λ is sufficiently close to one, then the operator $F(\sigma)$ is differentiable, and the operator $F'(\sigma)$ is Lipschitz continuous with respect to the topology of $H^1(\Omega)$.*

4 Applications to smoothness/sparsity regularization

Now we apply the analytical results of the previous sections to investigate several Tikhonov functionals for the EIT inverse problem. We focus on the continuum model. The complete electrode model can be treated analogously. The penalties of interest include smoothness and sparsity penalties. The former has been very popular, while the latter has demonstrated its potential only recently [32, 22].

The purpose of using a priori information such as smoothness or sparsity is to counter insufficient amount of information contained in the data as well as ill-posed nature of the problem. We are interested in reconstructing conductivities σ that away from a known background σ_0 are smooth or sparse. Let $\vartheta = \sigma - \sigma_0$ be the inclusions/inhomogeneities. Here the background σ_0 can be arbitrary, e.g., discontinuous. The setting we are going to use for ϑ is a Hilbert space $H_0^1(\Omega')$, i.e., $\vartheta \in H_0^1(\Omega')$, and we assume that the space $H_0^1(\Omega')$ is equipped with an orthonormal basis $\{\psi_k\}$. Then on the sequence $\{\langle\vartheta,\psi_k\rangle\}$ of expansion coefficients, we endow ℓ_r norms, i.e.,

$$\|\vartheta\|_{\ell_r}^r = \sum_{k=1}^\infty |\langle\vartheta,\psi_k\rangle|^r.$$

We consider the following penalty term $R_r(\vartheta)$

$$R_r(\vartheta) = \tfrac{1}{r}\|\vartheta\|_{\ell_r}^r \quad 1 \le r \le 2.$$

First, we observe that the penalty $R_r(\vartheta)$ is convex and weakly lower semi-continuous [7]. Second, the choice $r = 2$ reproduces the classical smoothness penalty, i.e., $R_2(\vartheta) = \frac{1}{2}\|\vartheta\|^2_{H^1(\Omega')}$ in view of the norm equivalence, which is one of most widely used penalties since the inaugural work [53], see [55, 51] for applications in EIT. Third, the choice $r \in [1, 2)$ is motivated by sparsity constraint [14]. Here ϑ is assumed to have a sparse representation in the basis $\{\psi_k\}$, i.e., only finitely many coefficients $\{\langle\vartheta,\psi_k\rangle\}$ are nonzero. It is widely accepted that sparsity may be promoted via an $R_r(r \in [1, 2))$, prominently R_1, penalty on expansion coefficients. Therefore, by considering an $R_r(r \in [1, 2])$ penalty, we treat smoothness/sparsity penalty in a unified way.

We study the linearized and fully nonlinear models separately, by capitalizing on recent progress on nonsmooth regularization [8, 46, 50, 27, 5, 41, 23, 45, 7]. Throughout this section, we assume Corollary 2.1 holds.

4.1 Linearized model

Although the EIT inverse problem is inherently nonlinear, a linearized model has been popular [56, 57, 11, 49, 40, 26]. This is partly attributed to the fact that there are diverse sources, possibly significant, of model uncertainties, in, e.g., geometry and boundary conditions. However, the analysis of such linearization procedure lags far behind, and many basic questions about the validity of the procedure remains unaddressed. Recently [26] shows that such a procedure preserves the outer support of the inclusions.

The linearized model consists of approximately solving the following operator equation

$$F'(\sigma_0)\vartheta + F(\sigma_0) - \phi^\delta = 0. \tag{7}$$

A first remark concerning the linearized problem (7) is as follows.

Remark 4.1. *By Lemma 2.4 and Corollary 2.1, the linear operator $F'(\sigma) : H_0^1(\Omega') \to \tilde{H}^1(\Omega)$ is bounded, and by trace theorem [20], i.e., $\tilde{H}^1(\Omega)$ embeds compactly into $L_2(\Gamma)$, the mapping $F'(\sigma) : H_0^1(\Omega') \to L_2(\Gamma)$ is thus bounded and compact. Consequently, the linearized equation (7) is ill-posed. The analysis developed here does not cover the TV penalty, e.g. [17], due to the fact the space of bounded variation $BV(\Omega')$ only embeds into $L_{\frac{d}{d-1}}(\Omega')$ [20], while the boundedness of $F'(\sigma) : L_{\frac{d}{d-1}}(\Omega') \to L_2(\Gamma)$ $(d = 2, 3)$ is yet to be established.*

Remark 4.2. *The space $\ell_r(r \in [1, 2))$ is a subspace of ℓ_2, and thus the R_r-penalty enforces a stronger penalization than $H^1(\Omega')$-penalty.*

According to the above remarks the linear operator equation (7) is ill-posed in the sense of Hadamard. For its stable numerical solution, typically Tikhonov regularization is applied

$$\Psi_{\mathrm{L}}(\vartheta) = \tfrac{1}{2}\|F'(\sigma_0)\vartheta - \tilde{\phi}^\delta\|^2_{L_2(\Gamma)} + \alpha R_r(\vartheta),$$

where $\tilde{\phi}^\delta = \phi^\delta - F(\sigma_0)$ denotes the linearized noisy data, and α is a scalar compromising the two terms.

A first question regarding any mathematical formulation is its well-posedness. By the results in Section 2 and Remark 4.1, we have the following existence and stability results. It addresses a linear inverse problem, hence it follows directly from general results in [14].

Theorem 4.1. *There exists at least one minimizer ϑ_α^δ to the functional Ψ_{L}. Let $\{\tilde{\phi}^n\} \subset L_2(\Gamma)$ be a sequence of noisy data converging to $\tilde{\phi}^\delta$, and ϑ^n be a minimizer to Ψ_{L} with $\tilde{\phi}^n$ in place of $\tilde{\phi}^\delta$. Then the sequence $\{\vartheta^n\}$ has a subsequence converging in $H^1(\Omega')$ to a minimizer of Ψ_{L}.*

To state a consistency result, we first recall the concept of an R_r-minimizing solution $\vartheta^\dagger$, i.e.,

$$\vartheta^\dagger = \arg\min_{\vartheta \in \mathcal{S}} R_r(\vartheta),$$

where the set $\mathcal{S} = \{\vartheta \in H_0^1(\Omega') : \|F'(\sigma_0)\vartheta - \tilde{\phi}^\dagger\|_{L_2(\Gamma)} = 0\}$.

The formulation Ψ_{L} employs a linearized model, which represents the full nonlinear model only approximately. Hence, it is not obvious that an exact solution for noiseless data exists, i.e., the data $\tilde{\phi}^\dagger = \phi^\dagger - F(\sigma_0)$ may lie beyond the range of $F'(\sigma_0)$, which would result in an empty set $\mathcal{S}$. In this case $\mathcal{S}$ would need to be defined as the set of parameters ϑ attaining a minimum of the residual. In order to avoid this complication we a priori assume the existence of a solution of the linearized problem. We note that $\vartheta^\dagger$ is generally different from the true inhomogeneity $\sigma^\dagger - \sigma_0$.

Now we can state the following consistency result [14].

Theorem 4.2. *Assume that there exists an R_r-minimizing solution. If the parameter $\alpha = \alpha(\delta)$ satisfies $\lim_{\delta\to 0} \alpha(\delta) = 0$ and $\lim_{\delta\to 0} \frac{\delta^2}{\alpha(\delta)} = 0$, then the sequence of minimizers $\{\vartheta_\alpha^\delta\}$ has a convergent subsequence in $H^1(\Omega')$ to an R_r-minimizing solution $\vartheta^\dagger$ as $\delta \to 0$. Further, if $\vartheta^\dagger$ is unique, then the whole sequence converges.*

In order to obtain quantitative estimates for the minimizer ϑ_α^δ, a source condition on the solution $\vartheta^\dagger$, is required, and we refer to [28] for an up-to-date account of such conditions. To this end, we need the adjoint of the operator $F'(\sigma)$ with respect to the $H_0^1(\Omega')$ inner product.

Theorem 4.3. *The adjoint of the operator $F'(\sigma) : H_0^1(\Omega') \to L_2(\Gamma)$ is given by*

$$\begin{aligned}(F'(\sigma))^* : L_2(\Gamma) &\to H_0^1(\Omega'),\\ f &\mapsto \tilde{u},\end{aligned}$$

where $\tilde{u} \in H_0^1(\Omega')$ solves $-\nabla^2\tilde{u} + \tilde{u} = -\nabla F(\sigma) \cdot \nabla w$ in Ω', and $w \in \tilde{H}^1(\Omega)$ is the solution to the adjoint problem

$$\int_\Omega \sigma \nabla w \cdot \nabla v dx = \int_\Gamma f v ds \quad \forall v \in \tilde{H}^1(\Omega).$$

Proof. For any $\vartheta \in H_0^1(\Omega')$, we have by Theorem 2.4

$$\begin{aligned}\langle f, F'(\sigma)\vartheta\rangle_{L_2(\Gamma)} &= \langle \vartheta, -\nabla F(\sigma) \cdot \nabla w\rangle_{L_2(\Omega')}\\ &= \langle i\vartheta, -\nabla F(\sigma) \cdot \nabla w\rangle_{L_2(\Omega')}\\ &= \langle \vartheta, i^*(-\nabla F(\sigma) \cdot \nabla w)\rangle_{H^1(\Omega')},\end{aligned}$$

where i is the embedding operator from $H_0^1(\Omega')$ into $L_2(\Omega')$. The assertion follows directly from the expression for the adjoint i^* of the embedding operator, c.f., [44, Cor. 2.3]. $\square$

We shall measure the error in Bregman distance [8]. We denote by $\partial R_r(\vartheta)$ the subdifferential of the convex functional R_r at ϑ, i.e., $\partial R_r(\vartheta) = \{\xi \in H_0^1(\Omega') : R_r(\vartheta') - R_r(\vartheta) \geq \langle \xi, \vartheta' - \vartheta\rangle \ \forall \vartheta' \in H_0^1(\Omega')\}$, and define the Bregman distance $d_\xi(\vartheta, \vartheta^\dagger)$ between ϑ and $\vartheta^\dagger$ relative to any $\xi \in \partial R_r(\vartheta^\dagger)$ by

$$d_\xi(\vartheta, \vartheta^\dagger) = R_r(\vartheta) - R_r(\vartheta^\dagger) - \langle \xi, \vartheta - \vartheta^\dagger\rangle.$$

Now we can state a first estimate for the linearized model, which follows directly from the general theory [8].

Theorem 4.4. *Let $\vartheta^\dagger$ be an R_r-minimizing solution, and assume that it satisfies the following source condition: there exists a $w \in L_2(\Gamma)$ such that $(F'(\sigma_0))^* w = \xi \in \partial R_r(\vartheta^\dagger)$. Then for a parameter choice rule $\alpha \sim \delta$, there hold*

$$d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \leq C\delta \quad \text{and} \quad \|F'(\sigma_0)\vartheta_\alpha^\delta - \tilde{\phi}^\delta\|_{L_2(\Gamma)} \leq C\delta.$$

In case of R_2, Theorem 4.4 gives an estimate in $H^1(\Omega')$ for the conventional smoothness penalty. We remark that Theorem 4.4 actually gives an estimate in the $H^1(\Omega')$-norm for any $r \in (1, 2]$. To see this, we recall the following inequality for Bregman distance $d_\xi(\vartheta, \vartheta^\dagger)$ [5, Lem. 2.7] [23, Lem. 10].

Lemma 4.1. *Let $r \in (1, 2]$. There exists a constant $c_r > 0$ depending only on r such that*

$$d_\xi(\vartheta', \vartheta) := R_r(\vartheta') - R_r(\vartheta) - \langle \xi, \vartheta' - \vartheta\rangle \geq \frac{c_r \|\vartheta' - \vartheta\|_{H^1(\Omega')}^2}{3 + 2R_r(\vartheta) + R_r(\vartheta')}$$

for all $\vartheta', \vartheta \in \mathrm{dom}(R_r)$ for which $\partial R_r(\vartheta) \neq \emptyset$.

Hence the estimate in Theorem 4.4 implies a convergence rate of $\mathcal{O}(\delta^{\frac{1}{2}})$ in the $H^1(\Omega')$-norm. However, the interesting case R_1 is not covered. This can be remedied by imposing extra conditions [41, 23].

Lemma 4.2. *Let $r \in [1, 2)$ and the solution $\vartheta^\dagger$ have a finite support $\mathbb{K}$ with respect to $\{\psi_k\}$. Let $\vartheta^\dagger$ satisfy the following source condition: there exists a $w \in L_2(\Gamma)$ such that $(F'(\sigma_0))^* w = \xi \in \partial R_r(\vartheta^\dagger)$, and $F'(\sigma_0)$ be injective on $\{\psi_k, k \in \mathbb{K}\}$. Then for any $\xi \in \partial R_r(\vartheta^\dagger)$*

$$\|\vartheta - \vartheta^\dagger\|_{H^1(\Omega')}^r \leq C_1 \|F'(\sigma_0)(\vartheta - \vartheta^\dagger)\|_{L_2(\Gamma)}^r + C_2 d_\xi(\vartheta, \vartheta^\dagger).$$

Proof. The proof can be found in [23, Thms. 14 and 15] for the cases $r > 1$ and $r = 1$, respectively. $\square$

Now we can show an enhanced convergence rate of order $\mathcal{O}(\delta^{\frac{1}{r}})$.

Theorem 4.5. *Let $r \in [1,2)$ and assume that the conditions in Theorem 4.4 hold. Further, let $\vartheta^\dagger$ have a finite support $\mathbb{K}$ with respect to $\{\psi_k\}$, and the operator $F'(\sigma_0)$ be injective on $\{\psi_k, k \in \mathbb{K}\}$. Then for a choice rule $\alpha \sim \delta$ there holds*

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} \leq C\delta^{\frac{1}{r}}.$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^r &\leq C_1\|F'(\sigma_0)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)}^r + C_2 d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \\ &\leq C_1\left(\|F'(\sigma_0)\vartheta_\alpha^\delta - \tilde{\phi}^\delta\|_{L_2(\Gamma)} + \|F'(\sigma_0)\vartheta^\dagger - \tilde{\phi}^\delta\|_{L_2(\Gamma)}\right)^r + C_2 d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger)\end{aligned}$$

Now the second assertion follows from the choice $\alpha \sim \delta$ and Theorem 4.4. ☐

Remark 4.3. *The injectivity of the operator $F'(\sigma_0)$ on the finite-dimensional subspace $\{\psi_k, k \in \mathbb{K}\}$ is crucial for deriving the enhanced convergence rate. This property was shown for piecewise polynomial/analytic conductivity distributions in case of full measurements, i.e., the operator $F(\sigma)$ is the Neumann-to-Dirichlet map [39, 26]. Therefore, the estimate in Theorem 4.5 holds with the further restriction on the basis $\{\psi_k\}$ of being piecewise analytic.*

Remark 4.4. *One can show that under certain conditions, with α chosen by the discrepancy principle, i.e., α satisfies $\|F'(\sigma_0)\vartheta_\alpha^\delta - \tilde{\phi}^\delta\| = c\delta$ $(c \geq 1)$, the solution ϑ_α^δ also converges, and analogous estimates as in Theorems 4.4 and 4.5 hold, see, e.g., [33]. The discrepancy principle is useful if an estimate of the noise level δ is known.*

4.2 Nonlinear model

Now we turn to the full nonlinear model. Some theoretical studies concerning the nonlinear EIT model have been carried out in [48] for a Mumford-Shah penalty and in [47] for a total variation penalty. Here we consider the R_r-penalty term, which covers both conventional H^1- and sparsity penalty, i.e.,

$$\Psi(\vartheta) = \tfrac{1}{2}\|F(\sigma) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha R_r(\vartheta), \tag{8}$$

and we shall again denote the minimizer by ϑ_α^δ. Such penalties have been treated under suitable assumptions for general non-linear operators, e.g., in [27]. We refer to these results whenever appropriate.

We begin with the following existence and stability result, which is a consequence of the analytical results in Section 2.

Theorem 4.6. *There exists at least one minimizer ϑ_α^δ to the functional $\Psi(\sigma)$ on the admissible set $\mathcal{A}$. Let $\{\phi^n\} \subset L_2(\Gamma)$ be a sequence of noisy data converging to ϕ^δ, and ϑ^n be a minimizer to Ψ with ϕ^n in place of ϕ^δ. Then the sequence $\{\vartheta^n\}$ has a subsequence converging in $H^1(\Omega')$ to a minimizer of Ψ. Moreover, if the parameter $\alpha = \alpha(\delta)$ satisfies $\lim_{\delta\to 0}\alpha(\delta) = 0$ and $\lim_{\delta\to 0}\frac{\delta^2}{\alpha(\delta)} = 0$, then the sequence of minimizers $\{\vartheta_\alpha^\delta\}$ has a subsequence converging in $H^1(\Omega')$ to an R_r-minimizing solution $\vartheta^\dagger$ as $\delta \to 0$. Furthermore, if $\vartheta^\dagger$ is unique, then the whole sequence converges.*

Proof. We only sketch the existence proof. The nonnegativity of Ψ implies the existence of a minimizing sequence $\{\vartheta^n\} \subset \mathcal{A}$, for which $\{R_r(\vartheta^n)\}$ is uniformly bounded. From the inequality $\|\vartheta^n\|_{\ell_2} \leq \|\vartheta^n\|_{\ell_r} \leq C$ for $r \leq 2$, we deduce uniform boundedness of $\{\vartheta^n\}$ in $H^1(\Omega')$. Therefore, there exists a subsequence of $\{\vartheta^n\}$, also denoted by $\{\vartheta^n\}$, and some $\vartheta^* \in H_0^1(\Omega)$, such that $\vartheta^n \to \vartheta^*$ weakly in $H^1(\Omega')$. By Kondrashov embedding theorem [20], it converges strongly in $L_p(\Omega')$ for any $p < 6$ in case of $d = 2, 3$. Proposition 2.1 implies $\lim_{n\to\infty} J(\sigma^n) = J(\sigma^*)$, from which and weak lower semicontinuity of $R_r(\vartheta)$ follows the desired assertion. The rest follows from the general theory of sparsity constrained nonlinear inverse problems, see e.g., [27]. ☐

Next we state a first estimate for the minimizer ϑ_α^δ. The proof is quite standard, see e.g., [19, 27], but we include its proof for completeness.

Theorem 4.7 ($1 < r \le 2$)**.** *Let $r \in (1,2]$, and the solution $\vartheta^\dagger$ satisfy the source condition: there exists a $w \in L_2(\Gamma)$ such that the $(F'(\sigma^\dagger))^* w = \xi \in \partial R_r(\vartheta^\dagger)$ with $\|w\|_{L_2(\Gamma)}$ sufficiently small. Then for a choice rule $\alpha \sim \delta$, there holds*

$$d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \le C\delta \quad \textit{and} \quad \|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)} \le C\delta.$$

Proof. The minimizing property of ϑ_α^δ implies

$$\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha R_r(\vartheta_\alpha^\delta) \le \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha R_r(\vartheta^\dagger).$$

This yields $R_r(\vartheta_\alpha^\delta) \le C + R_r(\vartheta^\dagger)$ for a choice rule $\alpha \sim \delta$. Appealing to the source condition and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \le& \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 - \alpha\langle \xi, \vartheta_\alpha^\delta - \vartheta^\dagger\rangle \\
\le& \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha\|w\|_{L_2(\Gamma)}\|F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\|_{L_2(\Gamma)} \\
&+ \alpha\|w\|_{L_2(\Gamma)}\|\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\|_{L_2(\Gamma)},
\end{aligned} \tag{9}$$

where the linearization error $\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)$ is defined as

$$\theta(\vartheta_\alpha^\delta, \vartheta^\dagger) = F(\sigma_\alpha^\delta) - F(\sigma^\dagger) - F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger). \tag{10}$$

By the Lipschitz continuity of $F'(\sigma)$ in Corollary 2.1, we have

$$\|\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\|_{L_2(\Gamma)} \le \tfrac{L}{2}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^2. \tag{11}$$

With the help of triangle inequality and Young's inequality, we deduce

$$\alpha\|w\|_{L_2(\Gamma)}\|F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\|_{L_2(\Gamma)} \le \alpha\|w\|_{L_2(\Gamma)}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)} + \tfrac{1}{2}\alpha^2\|w\|_{L_2(\Gamma)}^2 + \tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2.$$

In view of the preceding three inequalities, we arrive at

$$\alpha\left[d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) - \tfrac{L}{2}\|w\|_{L_2(\Gamma)}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^2\right] \le \tfrac{1}{2}\left(\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)} + \alpha\|w\|_{L_2(\Gamma)}\right)^2.$$

The first assertion follows from this inequality, Lemma 4.1 and the choice of α. Next we estimate the term $\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}$. From inequality (9), we have

$$\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \le \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 - \alpha\langle w, F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\rangle + \alpha\langle w, \theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\rangle,$$

which upon completing the squares gives

$$\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta + \alpha w\|_{L_2(\Gamma)}^2 + \alpha d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \le \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta + \alpha w\|_{L_2(\Gamma)}^2 + \alpha\|w\|_{L_2(\Gamma)}\|\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\|_{L_2(\Gamma)}.$$

The desired assertion on $\|F(\sigma) - \phi^\delta\|_{L_2(\Gamma)}$ follows from this and the choice $\alpha \sim \delta$. □

Remark 4.5. *An inspection of the proof indicates that restriction on the size of $\|w\|_{L_2(\Gamma)}$ depends on three quantities: the Lipschitz constant of $F'(\sigma)$, the exponent r and $R_r(\vartheta^\dagger)$. In case of $r = 2$, the Bregman distance $d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger)$ reduces to $\frac{1}{2}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^2$, and thus the condition can be explicitly written as $L\|w\|_{L_2(\Gamma)} < 1$.*

By Lemma 4.1, Theorem 4.7 gives a convergence rate $\mathcal{O}(\delta^{\frac{1}{2}})$ in $H^1(\Omega')$. Additional conditions on $\vartheta^\dagger$ can enhance the convergence rate from $\mathcal{O}(\delta^{\frac{1}{2}})$ to $\mathcal{O}(\delta^{\frac{1}{r}})$.

Theorem 4.8 $(1 < r < 2)$**.** *Let the conditions in Theorem 4.7 hold, $\vartheta^\dagger$ have a finite support $\mathbb{K}$ with respect to $\{\psi_k\}$, and the operator $F'(\sigma^\dagger)$ be injective on $\{\psi_k, k\in\mathbb{K}\}$. Then for a choice rule $\alpha\sim\delta$ there holds*

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} \leq C\delta^{\frac{1}{r}}.$$

Proof. By Lemma 4.2, we have

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^r \leq C_1\|F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)}^r + C_2 d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger).$$

We estimate the term $\|F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)}$ by noting (10) as follows

$$\begin{aligned}\|F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)} &\leq \|F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\|_{L_2(\Gamma)} + \|F(\sigma_\alpha^\delta) - F(\sigma^\dagger) - F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)}\\ &\leq \|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)} + \|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)} + \tfrac{L}{2}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^2.\end{aligned}$$

The term $\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}$ can be bounded as follows

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}^2 \leq C d_\xi(\vartheta_\alpha^\delta, \vartheta) \leq C\delta,$$

by Lemma 4.1 and Theorem 4.7. The conclusion follows from these estimates. □

Theorem 4.8 does not cover the case R_1. Nonetheless, an analogous estimate remains valid.

Theorem 4.9 $(r = 1)$**.** *Let the solution $\vartheta^\dagger$ be unique, satisfy the source condition: there exists a $w\in L_2(\Gamma)$ such that $(F'(\sigma^\dagger))^*w \in \partial R_1(\vartheta^\dagger)$ and have a finite support $\mathbb{K}$ with respect to $\{\psi_k\}$, and the operator $F'(\sigma^\dagger)$ be injective on $\{\psi_k, k\in\mathbb{K}\}$. Then for small δ and a choice rule $\alpha\sim\delta$, there holds*

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} \leq C\delta^{\frac{1}{2}}.$$

Proof. The minimizing property of ϑ_α^δ implies

$$\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha R_1(\vartheta_\alpha^\delta) \leq \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha R_1(\vartheta^\dagger),$$

which together with the choice $\alpha\sim\delta$ yields

$$\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)} \leq C_3\delta^{\frac{1}{2}}. \tag{12}$$

Appealing to the source condition $\xi = (F'(\sigma^\dagger))^*w \in \partial R_1(\vartheta^\dagger)$ and the definition of linearization error $\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)$, i.e., (10), we deduce

$$\begin{aligned}\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 + \alpha d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) &\leq \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 - \alpha\langle w, F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\rangle\\ &= \tfrac{1}{2}\|F(\sigma^\dagger) - \phi^\delta\|_{L_2(\Gamma)}^2 - \alpha\langle w, F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\rangle + \alpha\langle w, \theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\rangle.\end{aligned} \tag{13}$$

In particular, this together with the choice $\alpha\sim\delta$ implies

$$d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger) \leq C_4\delta - \langle w, F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\rangle + \langle w, \theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\rangle.$$

Next by Lemma 4.2, we have

$$\begin{aligned}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} &\leq C_1\|F'(\sigma^\dagger)(\vartheta_\alpha^\delta - \vartheta^\dagger)\|_{L_2(\Gamma)} + C_2 d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger)\\ &\leq C_1\left(\|F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\|_{L_2(\Gamma)} + \|\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\|_{L_2(\Gamma)}\right) + C_2 d_\xi(\vartheta_\alpha^\delta, \vartheta^\dagger).\end{aligned}$$

These two inequalities together with (11), Cauchy-Schwarz inequality and estimate (12) yield

$$\begin{aligned}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} \leq& C_2C_4\delta + (C_1 + C_2\|w\|_{L_2(\Gamma)})\|F(\sigma_\alpha^\delta) - F(\sigma^\dagger)\|_{L_2(\Gamma)}\\ &+ (C_1 + C_2\|w\|_{L_2(\Gamma)})\|\theta(\vartheta_\alpha^\delta, \vartheta^\dagger)\|_{L_2(\Gamma)}\\ \leq& C_2C_4\delta + (C_1 + C_2\|w\|_{L_2(\Gamma)})(\delta + C_3\delta^{\frac{1}{2}}) + (C_1 + C_2\|w\|_{L_2(\Gamma)})\tfrac{L}{2}\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{L_2(\Gamma)}^2.\end{aligned}$$

Upon letting $C_5 = (C_1 + C_2\|w\|_{L_2(\Gamma)})\frac{L}{2}$ and $C_6 = (C_1 + C_2\|w\|_{L_2(\Gamma)})(\delta^{\frac{1}{2}} + C_3) + C_2C_4\delta^{\frac{1}{2}}$, this gives $C_5t^2 - t + C_6\delta^{\frac{1}{2}} \geq 0$ for $t = \|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}$. For sufficiently small δ, we have $1 - 4C_5C_6\delta^{\frac{1}{2}} > 0$. Hence, the above quadratic polynomial in t has two distinct positive roots, and the inequality amounts to

$$t \geq \frac{1 + \sqrt{1 - 4C_5C_6\delta^{\frac{1}{2}}}}{2C_5} \quad \text{or} \quad t \leq \frac{1 - \sqrt{1 - 4C_5C_6\delta^{\frac{1}{2}}}}{2C_5}.$$

By virtue of the consistency result in Theorem 4.6, the latter case holds, i.e.,

$$\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} \leq \frac{1 - \sqrt{1 - 4C_5C_6\delta^{\frac{1}{2}}}}{2C_5} \leq 2C_6\delta^{\frac{1}{2}},$$

where we have utilized the elementary inequality $\sqrt{1-s} \geq 1-s$, $\forall s \in [0,1]$. This concludes the proof. $\square$

Remark 4.6. *The estimate* (12) *yields an upper bound for the discrepancy term, which can be improved by a bootstrap argument as follows: The estimate in Theorem 4.9 and applying the Cauchy-Schwarz inequality to* (13) *leads to*

$$\tfrac{1}{2}\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)}^2 \leq \tfrac{1}{2}\delta^2 + \alpha\|w\|_{L_2(\Gamma)}(\delta + C_3\delta^{1/2}) + \alpha\|w\|_{L_2(\Gamma)}\tfrac{L}{2}\delta$$

and hence $\|F(\sigma_\alpha^\delta) - \phi^\delta\|_{L_2(\Gamma)} \leq C\delta^{\frac{3}{4}}$.

This improved estimate for the discrepancy term can also be used to obtain an improved estimate for $\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')}$. *By repeating the arguments in the proof above we get an estimate of order* $\|\vartheta_\alpha^\delta - \vartheta^\dagger\|_{H^1(\Omega')} = \mathcal{O}(\delta^{\frac{3}{4}})$. *This bootstrap procedure can be repeated to derive convergence rate of order* $\mathcal{O}(\delta^s)$ *for any* $s < 1$.

5 Concluding remarks

In this paper we have presented an analysis of two electrical impedance tomography models, i.e., continuum model and complete electrode model, and the continuity and differentiability of the forward operator with respect to L_p norms are shown. The analytical results are applied to several regularization formulations with smoothness/ sparsity penalty for the linearized and nonlinear models, in particular the conventional H^1 penalty and the recent sparsity penalty. The existence of a minimizer, stability, consistency and convergence rate for these formulations are discussed.

There are several avenues for further research. Firstly, we have restricted our attention to Tikhonov regularization with an ℓ_r-penalty. Alternative approaches, e.g., iterative regularization methods such as Landweber and Gauss-Newton methods in Banach spaces [50, 5, 35], might also be justified using the presented analytical results. Secondly, the operator $F'(\sigma)$ deserves further attention, e.g. the solvability of the linearized equation. Finally, refined regularity for the forward model is of immense interest. The derivations herein utilize Meyers estimate, which relies only on the $L_\infty(\Omega)$ bound of the parameter σ, and the extra regularity on the conductivity σ, e.g., BV or H^1, might enable deriving more refined estimates.

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