DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Adaptive Wavelet Methods for Elliptic Stochastic Partial Differential Equations

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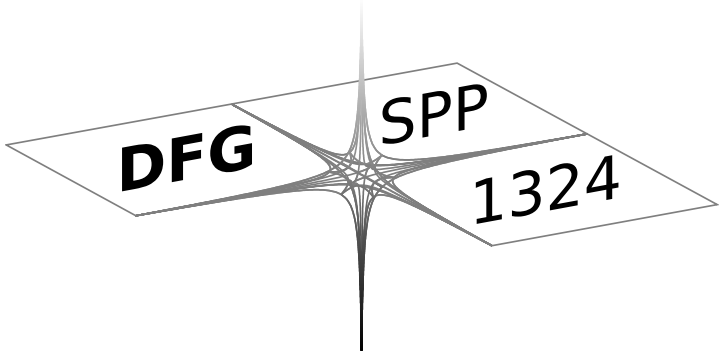
Preprint 77



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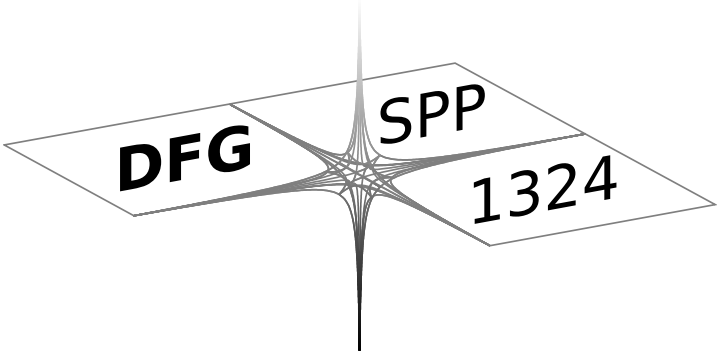

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ADAPTIVE WAVELET METHODS FOR ELLIPTIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

P. A. CIOICA, S. DAHLKE, N. DÖHRING, S. KINZEL, F. LINDNER,
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Abstract. We study the Besov regularity as well as linear and nonlinear approximation of random functions on bounded Lipschitz domains in $\mathbb{R}^d$. The random functions are given either (i) explicitly in terms of a wavelet expansion or (ii) as the solution of a Poisson equation with a right-hand side in terms of a wavelet expansion. In the case (ii) we derive an adaptive wavelet algorithm that achieves the nonlinear approximation rate at a computational cost that is proportional to the degrees of freedom. These results are matched by computational experiments.

1. Introduction

We study numerical algorithms for the Poisson equation

$$\begin{aligned} -\Delta U &= X \quad \text{in} \quad D, \\ U &= 0 \quad \text{on} \quad \partial D \end{aligned} \tag{1}$$

with random right-hand side X, and $D \subset \mathbb{R}^d$ a bounded Lipschitz domain. More precisely, X is a random function (random field) with realizations at least in $L_2(D)$, and we wish to approximate the realizations of the random function U in $H^1(D)$. We investigate a new stochastic model for X that provides an explicit control of the Besov regularity, and we analyse nonlinear approximation of both X and U. An average N-term approximation of the right-hand side X can be simulated efficiently, and the nonlinear approximation rate for the solution U is achieved by means of an adaptive wavelet algorithm. These asymptotic results are matched by numerical experiments.

Different numerical problems have been studied for Poisson equations, or more generally, for elliptic equations with a random right-hand side and/or a random diffusion coefficient. The computational task is to approximate either the realizations of the solution or at least their moments, and different techniques like stochastic finite element methods, sparse grids, or polynomial chaos decompositions are employed. A, by no means complete, list of papers includes [2, 12, 27, 36, 40, 47, 51, 55].

We construct efficient algorithms for nonlinear approximation of the random functions X and U. While nonlinear approximation methods are extensively studied in the deterministic case, see [25] and the references therein for details and a survey, much less is known for random functions. For the latter we refer to [9, 11], where wavelet methods are analysed, and to [13, 34] where free knot splines are used. In these papers the random functions are given explicitly, and the one-dimensional case $d = 1$ is studied. Stochastic

Mathematics Subject Classification (2010). 60H35, 65C30, 65T60, 41A25, Secondary: 41A63.

Key words and phrases. elliptic stochastic partial differential equation, wavelets, Besov regularity, approximation rates, nonlinear approximation, adaptive methods.

This work has been supported by the Deutsche Forschungsgemeinschaft (DFG, grants DA 360/13-1, RI 599/4-1, SCHI 419/5-1) and a doctoral scholarship of the Philipps-Universität Marburg.

differential equations, in general, yield implicitly given random functions, which holds true in particular for U in (1). For stochastic ordinary differential equations nonlinear approximation of the solution process is studied in [13, 43].

The random function X is defined in terms of a stochastic wavelet expansion

$$X = \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} Y_{j,k} Z'_{j,k} \psi_{j,k}. \tag{2}$$

Here $\{\psi_{j,k} : j \geq j_0, k \in \nabla_j\}$ is a wavelet Riesz basis for $L_2(D)$, where j denotes the scale parameter and ∇_j is a finite set with, in order of magnitude, 2^{jd} elements, and $Y_{j,k}$ and $Z'_{j,k}$ are independent random variables. In a slightly simplified version of the stochastic model, $Y_{j,k}$ is Bernoulli distributed with parameter $2^{-\beta jd}$ and $Z'_{j,k}$ is normally distributed with mean zero and variance $2^{-\alpha jd}$, where $0 \leq \beta \leq 1$ and $\alpha + \beta > 1$. Note that the sparsity of the expansion (2) depends monotonically on β. For $\beta = 0$, i.e., with no sparsity present, (2) is the Karhunen-Loève expansion of a Gaussian random function X if the wavelets form an orthonormal basis of $L_2(D)$. The stochastic model (2) was introduced and analysed in the context of Bayesian non-parametric regression in [1, 3].

Let us point to the main results. The random function X takes values in the Besov space $B^s_q(L_p(D))$ with probability one if and only if

$$s < d \cdot \left(\frac{\alpha - 1}{2} + \frac{\beta}{p} \right),$$

see Theorem 2.6. In [1, 3] the result was stated for $d = 1$ and $p, q \geq 1$. In particular, the smoothness of X along the scale of Sobolev spaces $H^s(D) = B^s_2(L_2(D))$ is determined by $\alpha + \beta$, and for $\beta > 0$ with decreasing $p \in]0, 2]$ the smoothness can get arbitrarily large.

We study different approximations $\widehat{X}$ of X with respect to the norm in $L_2(D)$; we always consider the average error $(\mathrm{E}\, \|X - \widehat{X}\|^2_{L_2(D)})^{1/2}$ for any approximation $\widehat{X}$. For the optimal linear approximation, i.e., for the approximation from an optimally chosen N-dimensional subspace of $L_2(D)$, the corresponding errors are asymptotically equivalent to $N^{-\varrho}$ with

$$\varrho = \frac{\alpha + \beta - 1}{2},$$

see Theorem 2.9. In contrast, for the best average N-term wavelet approximation we only require that the average number of non-zero wavelet coefficients is at most N. In this case the corresponding errors are at most of order $N^{-\varrho}$ with

$$\varrho = \frac{\alpha + \beta - 1}{2(1 - \beta)}$$

and $\beta < 1$, see Theorem 2.11. The best average N-term approximation is superior to optimal linear approximation if $\beta > 0$. The simulation of the respective average N-term approximation is possible at an average computational cost of order N, which is crucial in computational practice.

Considering the Poisson equation (1) with a right-hand side given by (2), the solution U of the Poisson equation is approximated with respect to the norm in $H^1(D)$. We consider the average error $(\mathrm{E}\, \|U - \widehat{U}\|^2_{H^1(D)})^{1/2}$ for any approximation $\widehat{U}$. Here the space $H^1(D)$ is the natural choice, since its norm is equivalent to the energy norm and the convergence analysis of adaptive wavelet algorithms relies on this norm. We study the N-term wavelet approximation under different assumptions on the domain D, and we establish upper

bounds of the form $N^{-(\varrho-\varepsilon)}$, which hold for every $\varepsilon > 0$. For any bounded Lipschitz domain D in dimension $d = 2$ or 3 we obtain

$$\varrho = \min\left(\frac{1}{2(d-1)}, \frac{\alpha+\beta-1}{6} + \frac{2}{3d}\right),$$

see Theorem 3.1. Regardless of the smoothness of X we have $\varrho \le 1/(2(d-1))$. On the other hand, uniform approximation schemes can only achieve the order $N^{-1/(2d)}$ on general Lipschitz domains D, and we always have $\varrho > 1/(2d)$. For more specific domains we fully benefit from the smoothness of the right-hand side. First,

$$\varrho = \frac{\alpha+\beta}{2}$$

if D is a simply connected polygonal domain in $\mathbb{R}^2$, see Theorem 3.5, and

$$\varrho = \frac{1}{1-\beta}\left(\frac{\alpha-1}{2} + \beta\right) + \frac{1}{d}$$

for bounded C^∞-domains $D \subset \mathbb{R}^d$, see Theorem 3.6.

These rates for the best N-term approximation of U are actually achieved by suitable adaptive wavelet algorithms, which have been developed for deterministic elliptic PDEs, see Section 4. Those algorithms converge for a large class of operators, including operators of negative order, and they are asymptotically optimal in the sense that they realize the optimal order of convergence while the computational cost is proportional to the degrees of freedom, see [7, 8, 15]. Moreover, the algorithmic approach can be extended to wavelet frames, i.e., to redundant wavelet systems, which are much easier to construct than wavelet bases on general domains, see [18, 44].

Numerical experiments are presented in Section 5. At first we illustrate some features of the stochastic model (2) for X, subsequently we determine empirical rates of convergence for adaptive and uniform approximation of the solution U to the Poisson equation (1) in dimension $d = 1$. It turns out that the empirical rates fit very well to the asymptotic results, and we observe superiority of the adaptive scheme already for moderate accuracies.

Throughout the paper we use the following notation. We write $A \preceq B$ for mappings $A, B : M \to [0, \infty]$ on any set M, if there exists a constant $c \in]0, \infty[$ such that $A(m) \le cB(m)$ holds for every $m \in M$. Furthermore $A \asymp B$ means $A \preceq B$ and $B \preceq A$. In the sequel the constants may depend on

- the domain D, the wavelet basis $\{\psi_{j,k} : j \ge j_0, k \in \nabla_j\}$, and the orthonormal basis $\{e_{j,k} : j \ge j_0, k \in \nabla_j\}$ employed in the proof of Theorem 2.9,
- the parameters s, p, and q of the Besov space,
- the parameters α, β, γ, C_1, and C_2 of the random function, and
- the parameter ε in Theorems 2.10, 3.1, 3.5, and 3.6.

The paper is organized as follows. In Sections 2 and 3 we study linear and nonlinear approximations of X and U, respectively; to this end we analyse the Besov regularity of these random functions. In Section 4 we explain how to achieve the nonlinear approximation rate by means of adaptive wavelet algorithms. In Section 5 we present numerical experiments to complement the asymptotic error analysis.

2. A Class of Random Functions in Besov Spaces

In this section we discuss linear and nonlinear approximations as well as the Besov regularity of *random functions* $X : \Omega \to L_2(D)$ on a bounded Lipschitz domain $D \subset \mathbb{R}^d$. The random functions are defined in terms of wavelet expansions according to a stochastic model that provides an explicit control for the Besov regularity and, in particular, induces sparsity of the wavelet coefficients. In the context of Bayesian non-parametric regression this model was introduced and analysed in [1] and generalized in [3] in the case $D = [0, 1]$ for Besov spaces with parameters $p, q \geq 1$.

2.1. **The Basic Wavelet Setting.** We briefly state the wavelet setting as far as it is needed for our purposes. In general, a *wavelet basis* $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ is a basis for an L_2-space with specific properties outlined below. The indices $\lambda \in \nabla$ typically encode several types of information, namely the *scale*, often denoted by $|\lambda|$, the *spatial location*, and also the *type* of the wavelet. For instance, on the real line, $|\lambda| = j \in \mathbb{Z}$ denotes the dyadic refinement level and $2^{-j}k$ with $k \in \mathbb{Z}$ stands for the location of the wavelet.

We will ignore any explicit dependence on the type of the wavelet from now on, since this only produces additional constants. Hence, we frequently use $\lambda = (j, k)$ and

$$\nabla = \{(j, k) : j \geq j_0, k \in \nabla_j\},$$

where ∇_j is some countable index set and $|(j, k)| = j$. Moreover, $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \nabla\}$ denotes the *dual wavelet basis*, which is biorthogonal to Ψ, i.e.,

$$\langle \psi_\lambda, \tilde{\psi}_{\lambda'} \rangle_{L_2(D)} = \delta_{\lambda,\lambda'}.$$

We assume that the domain D under consideration enables us to construct a wavelet basis Ψ with the following properties:

(A1) the wavelets form a *Riesz basis* for $L_2(D)$;

(A2) the *cardinalities* of the index sets ∇_j satisfy

$$\#\nabla_j \asymp 2^{jd};$$

(A3) the wavelets are *local* in the sense that

$$\operatorname{diam}(\operatorname{supp} \psi_\lambda) \asymp 2^{-|\lambda|};$$

(A4) the wavelets satisfy the *cancellation property*

$$|\langle v, \psi_\lambda \rangle_{L_2(D)}| \preceq 2^{-|\lambda|(\frac{d}{2}+\widetilde{m})} |v|_{W^{\widetilde{m}}(L_\infty(\operatorname{supp} \psi_\lambda))}$$

for $|\lambda| > j_0$ with some parameter $\widetilde{m} \in \mathbb{N}$;

(A5) the wavelet basis induces characterizations of *Besov spaces* $B^s_q(L_p(D))$ of the form

$$\|v\|_{B^s_q(L_p(D))} \asymp \left(\sum_{j=j_0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \left(\sum_{k \in \nabla_j} |\langle v, \tilde{\psi}_{j,k} \rangle_{L_2(D)}|^p \right)^{q/p} \right)^{1/q},$$

for $0 < p, q < \infty$ and all s with $d(\frac{1}{p} - 1)_+ < s < s_1$ for some parameter $s_1 > 0$.

In (A5) the upper bound s_1 depends, in particular, on the smoothness and the approximation properties of the wavelet basis.

For Section 3–5, we furthermore assume that the dual basis $\tilde{\Psi}$ satisfies (A1)–(A5) with possibly different parameters in (A4) and (A5). Using the fact that $B^s_2(L_2(D)) = H^s(D)$

and by exploiting the norm equivalence (A5), a simple rescaling of Ψ immediately yields a Riesz basis for $H^s(D)$ with $0 < s < s_1$. In Sections 4 and 5 we actually assume that the primal wavelet basis Ψ characterizes the Sobolev spaces $H^s_0(D)$ with homogeneous Dirichlet boundary conditions, instead of $H^s(D)$. Suitable constructions of wavelets on domains can be found in [4, 10, 22–24, 35, 38], and we also refer to [6] for a detailed discussion.

2.2. **The Stochastic Model.** The construction of the random function $X : \Omega \to L_2(D)$ primarily depends on the parameters

$$\alpha > 0, \qquad 0 \le \beta \le 1, \qquad \gamma \in \mathbb{R}$$

with

$$\alpha + \beta > 1.$$

Additionally, let $C_1, C_2 > 0$. The stochastic model is based on independent random variables $Y_{j,k}$ and $Z_{j,k}$ for $j \ge j_0$ and $k \in \nabla_j$ on some probability space $(\Omega, \mathcal{A}, P)$. The variables $Y_{j,k}$ are Bernoulli distributed with parameter

$$p_j = \min\left(1, C_1 2^{-\beta j d}\right), \quad \text{with} \quad P(Y_{j,k} = 1) = p_j \text{ and } P(Y_{j,k} = 0) = 1 - p_j.$$

The variables $Z_{j,k}$ are $N(0,1)$-distributed and, in order to rescale their variances, we put

$$\sigma_j^2 = C_2 j^{\gamma d} 2^{-\alpha j d}.$$

Since

$$\mathrm{E}\Big(\sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} \sigma_j^2 Y_{j,k}^2 Z_{j,k}^2\Big) = \sum_{j=j_0}^{\infty} \#\nabla_j \sigma_j^2 p_j \asymp \sum_{j=j_0}^{\infty} j^{\gamma d} 2^{-(\alpha+\beta-1) j d} < \infty$$

by (A2), we use (A1) to conclude that

$$X = \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} \sigma_j Y_{j,k} Z_{j,k} \psi_{j,k} \tag{3}$$

converges P-a.s. in $L_2(D)$.

Remark 2.1. Let $\xi, \zeta \in L_2(D)$. We have $\mathrm{E}(\langle \xi, X \rangle_{L_2(D)}) = 0$, i.e., X is a *mean zero* random function. Moreover,

$$\mathrm{E}(\langle \xi, X \rangle_{L_2(D)} \langle \zeta, X \rangle_{L_2(D)}) = \sum_{j=j_0}^{\infty} \sigma_j^2 p_j \sum_{k \in \nabla_j} \langle \xi, \psi_{j,k} \rangle_{L_2(D)} \langle \zeta, \psi_{j,k} \rangle_{L_2(D)}.$$

Using the dual basis we obtain

$$Q\xi = \sum_{j=j_0}^{\infty} \sigma_j^2 p_j \sum_{k \in \nabla_j} \langle \xi, \tilde{\psi}_{j,k} \rangle_{L_2(D)} \psi_{j,k}$$

for the *covariance operator* Q associated with X. If Ψ is an orthonormal basis, then (3) is the *Karhunen-Loève* decomposition of X. Note that X is *Gaussian* if and only if $p_j = 1$ for all $j \ge j_0$, i.e., $\beta = 0$ and $C_1 \ge 1$.

2.3. **Besov Regularity.** In the following we establish a necessary and sufficient condition for $X \in B^s_q(L_p(D))$ to hold with probability one. Consider the random variables

$$S_{j,p} = \sum_{k \in \nabla_j} Y_{j,k} |Z_{j,k}|^p, \quad j \geq j_0,$$

which form an independent sequence for every fixed $0 < p < \infty$. Furthermore, we use ν_p to denote the p-th absolute moment of the standard normal distribution.

Lemma 2.2. *Let $X_{n,\rho}$ be binomially distributed with parameters $n \in \mathbb{N}$ and $\rho \in [0,1]$. For every $r > 0$ there exists a constant $c > 0$ such that*

$$\mathrm{E}(X^r_{n,\rho}) \leq c\,(1 + (n\rho)^r)$$

for all n and ρ.

Proof. It suffices to consider $r \in \mathbb{N}$. The upper bound clearly holds for $r = 1$. For $r \geq 2$ we have $\mathrm{E}(X^r_{1,\rho}) = \rho \leq 1$. Moreover, for $n \geq 2$,

$$\begin{aligned}
\mathrm{E}(X^r_{n,\rho}) &= \sum_{k=1}^{n} k^r \binom{n}{k} \rho^k (1-\rho)^{n-k} \\
&= \sum_{k=0}^{n-1} (k+1)^{r-1} n \binom{n-1}{k} \rho^{k+1} (1-\rho)^{n-1-k} \\
&= n\rho\, \mathrm{E}(1 + X_{n-1,\rho})^{r-1}.
\end{aligned}$$

Inductively we obtain

$$\mathrm{E}(X^r_{n,\rho}) \preceq n\rho(1 + \mathrm{E}(X_{n-1,\rho})^{r-1}) \preceq n\rho + (n\rho)^r \preceq 1 + (n\rho)^r,$$

as claimed. □

Lemma 2.3. *Assume that $0 \leq \beta < 1$. Then*

$$\lim_{j \to \infty} \frac{S_{j,p}}{\#\nabla_j p_j} = \nu_p$$

holds with probability one, and

$$\sup_{j \geq j_0} \frac{\mathrm{E}(S^r_{j,p})}{(\#\nabla_j p_j)^r} < \infty$$

holds for every $r > 0$.

Proof. Clearly $\mathrm{E}(S_{j,p}) = \#\nabla_j p_j \nu_p$ and $\mathrm{Var}(S_{j,p}) = \#\nabla_j p_j (\nu_{2p} - p_j \nu_p^2)$. Hence

$$P(|S_{j,p}/(\#\nabla_j p_j) - \nu_p| \geq \varepsilon) \leq \varepsilon^{-2}\,(\#\nabla_j p_j)^{-1}(\nu_{2p} - p_j\nu_p^2) \preceq \varepsilon^{-2}\, 2^{-(1-\beta)jd}$$

follows from Chebyshev's inequality. It remains to apply the Borel-Cantelli Lemma to obtain the a.s. convergence of $S_{j,p}/(\#\nabla_j p_j)$.

Fix $r > 0$. By the equivalence of moments of Gaussian measures, see [49, p. 338], there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
\mathrm{E}\left(\left(\sum_{k \in \nabla_j} y_{j,k} |Z_{j,k}|^p\right)^r\right) &= \mathrm{E}\left(\left(\sum_{k \in \nabla_j} |y_{j,k} Z_{j,k}|^p\right)^r\right) \\
&\leq c_1 \left(\mathrm{E} \sum_{k \in \nabla_j} |y_{j,k} Z_{j,k}|^p\right)^r = c_1 \nu_p^r \left(\sum_{k \in \nabla_j} y_{j,k}\right)^r
\end{aligned}$$

for every $j \geq j_0$ and all $y_{j,k} \in \{0,1\}$. The sequences $(Y_{j,k})_k$ and $(Z_{j,k})_k$ are independent. This gives $\mathrm{E}(S_{j,p}^r) \leq c_1 \nu_p^r \, \mathrm{E}(S_{j,0}^r)$. Finally, there exists a constant $c_2 > 0$ such that $\mathrm{E}(S_{j,0}^r) \leq c_2 (\#\nabla_j p_j)^r$, see Lemma 2.2. □

Let μ denote any probability measure on the real line, and let $c > 0$. By $\mu^{*\ell}$ we denote the ℓ-fold convolution of μ. The compound Poisson distribution with intensity measure $c\mu$ is given by $\exp(-c) \sum_{\ell=0}^{\infty} \frac{c^\ell}{\ell!} \mu^{*\ell}$.

Lemma 2.4. *Assume that $\beta = 1$ and*

$$\lim_{j \to \infty} \#\nabla_j 2^{-jd} = C_0 \tag{4}$$

for some $C_0 > 0$. Let μ_p denote the distribution of $|Z_{j,k}|^p$, and let S_p be a compound Poisson random variable with intensity measure $C_0 \, C_1 \cdot \mu_p$. Then $(S_{j,p})_j$ converges in distribution to S_p, and

$$\sup_{j \geq j_0} \mathrm{E}(S_{j,p}^r) < \infty \tag{5}$$

holds for every $r > 0$.

Proof. Let Z be $N(0,1)$-distributed. The characteristic function φ_{S_p} of S_p is given by

$$\varphi_{S_p}(t) = \mathrm{E}(\exp(itS_p)) = \exp(C_0 \, C_1 (\varphi_{|Z|^p}(t) - 1)).$$

Furthermore, for the characteristic function $\varphi_{S_{j,p}}$ of $S_{j,p}$,

$$\varphi_{S_{j,p}}(t) = \Big(p_j \varphi_{|Z|^p}(t) + 1 - p_j \Big)^{\#\nabla_j} = \Big(1 + \frac{1}{\#\nabla_j} \cdot p_j \#\nabla_j \left(\varphi_{|Z|^p}(t) - 1 \right) \Big)^{\#\nabla_j}.$$

We use (4) to conclude that

$$\lim_{j \to \infty} \varphi_{S_{j,p}}(t) = \varphi_{S_p}(t),$$

which yields the convergence in distribution as claimed.

Suppose that $p \geq 1$. Then we take $c_1 > 0$ such that $z^{rp} \leq c_1 \exp(z)$ for every $z \geq 0$ and we put $c_2 = \mathrm{E}(\exp(|Z_{j,k}|))$ to obtain

$$\mathrm{E}(S_{j,p}^r) \leq \mathrm{E}(S_{j,1}^{rp}) \leq c_1 \, \mathrm{E}(\exp(S_{j,1})) = c_1 (1 + p_j (c_2 - 1))^{\#\nabla_j}.$$

Note that the upper bound converges to $c_1 \exp(C_0 C_1 (c_2 - 1))$. In the case $0 < p < 1$ we have $S_{j,p} \leq S_{j,0} + S_{j,1}$. Hence it remains to observe that $\sup_{j \geq j_0} \mathrm{E}(S_{j,0}^r) < \infty$, which follows from Lemma 2.2. □

Remark 2.5. In general, we only have (A2) instead of (4). For $\beta = 1$ the upper bound (5) remains valid in the general case, too. In all known constructions of wavelet bases on bounded domains, see, e.g., [4, 10, 22–24, 38], and also for wavelet frames [18, 44], the number $\#\nabla_j$ of wavelets per level $j > j_0$ is a constant multiple of 2^{jd}. For those kinds of bases, (4) trivially holds.

Theorem 2.6. *Suppose that $s > d \cdot (1/p - 1)_+$. We have $X \in B_q^s(L_p(D))$ with probability one if and only if*

$$s < d \cdot \left(\frac{\alpha - 1}{2} + \frac{\beta}{p} \right) \tag{6}$$

or

$$s = d \cdot \left(\frac{\alpha - 1}{2} + \frac{\beta}{p} \right) \qquad \text{and} \qquad q\gamma d < -2. \tag{7}$$

Furthermore,

$$\mathrm{E}\,\|X\|^q_{B^s_q(L_p(D))} < \infty$$

if (6) *or* (7) *is satisfied.*

Proof. Let

$$a_j = 2^{jq(s+d(\frac{1}{2}-\frac{1}{p}))}\sigma_j^q.$$

Since

$$\sum_{j=j_0}^{\infty} a_j (\#\nabla_j p_j)^{q/p} \asymp \sum_{j=j_0}^{\infty} j^{\frac{q\gamma d}{2}} 2^{\delta q j d}$$

with

$$\delta = \frac{s}{d} - \frac{\alpha-1}{2} - \frac{\beta}{p},$$

it follows that the condition (6) or (7) is equivalent to

$$\sum_{j=j_0}^{\infty} a_j (\#\nabla_j p_j)^{q/p} < \infty. \tag{8}$$

On the other hand, because of the characterization (A5), we have $X \in B^s_q(L_p(D))$ with probability one if and only if

$$\sum_{j=j_0}^{\infty} a_j S_{j,p}^{q/p} < \infty \quad P\text{-a.s.} \tag{9}$$

Therefore, it is enough to show the equivalence of (8) and (9).

In the case $0 \le \beta < 1$ this follows immediately from Lemma 2.3. Suppose that $\beta = 1$. Then $\#\nabla_j p_j \asymp 1$, so that (8) is equivalent to

$$\sum_{j=j_0}^{\infty} a_j < \infty. \tag{10}$$

On the other hand, since the random variables $a_j S_{j,p}$ are non-negative and independent, (9) is equivalent to

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left(\frac{a_j S_{j,p}^{q/p}}{1 + a_j S_{j,p}^{q/p}}\right) < \infty,$$

see [42, p. 363]. We use Lemma 2.4 and Remark 2.5 to conclude that (10) implies

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left(\frac{a_j S_{j,p}^{q/p}}{1 + a_j S_{j,p}^{q/p}}\right) \le \sum_{j=j_0}^{\infty} a_j\, \mathrm{E}(S_{j,p}^{q/p}) < \infty.$$

In the case $\sum_{j=j_0}^{\infty} a_j = \infty$ we put

$$c_1 = \inf_{j \ge j_0} P(S_{j,p} \ge 1).$$

We get $c_1 > 0$ from Lemma 2.4, and so

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left(\frac{a_j S_{j,p}^{q/p}}{1 + a_j S_{j,p}^{q/p}}\right) \ge c_1 \sum_{j=j_0}^{\infty} \frac{a_j}{1 + a_j} = \infty,$$

as claimed.

Finally, we consider the q-th moment of $\|X\|_{B^s_q(L_p(D))}$, provided that (8) is satisfied. We use (A5) and Lemma 2.3 for $0 \le \beta < 1$ as well as Lemma 2.4 and Remark 2.5 for $\beta = 1$ to derive

$$\mathrm{E}\,\|X\|^q_{B^s_q(L_p(D))} \preceq \sum_{j=j_0}^{\infty} a_j\,\mathrm{E}(S_{j,p}^{q/p}) \preceq \sum_{j=j_0}^{\infty} a_j (\#\nabla_j p_j)^{q/p} < \infty,$$

as claimed. □

For the conclusions in Theorem 2.6 to hold true we only assume (A1), (A2) and (A5). As a special case of Theorem 2.6, we consider the specific scale of Besov spaces $B^s_\tau(L_\tau(D))$ with

$$\frac{1}{\tau} = \frac{s-t}{d} + \frac{1}{2} \tag{11}$$

for $s > t \ge 0$, which determines the approximation order of best N-term wavelet approximation with respect to $H^t(D)$, see also Sections 2.4 and 3. In these sections average errors will be defined by second moments, and therefore we also study second moments of the norm in $B^s_\tau(L_\tau(D))$.

Corollary 2.7. *Suppose that*

$$0 \le t < d \cdot \frac{\alpha+\beta-1}{2}$$

and $s > t$ as well as

$$(1-\beta)\cdot s < d \cdot \frac{\alpha+\beta-1}{2} - \beta \cdot t.$$

Let τ be given by (11)*. Then $X \in B^s_\tau(L_\tau(D))$ holds with probability one, and*

$$\mathrm{E}\,\|X\|^2_{B^s_\tau(L_\tau(D))} < \infty.$$

Proof. Take $p = q = \tau$ and apply Theorem 2.6 to conclude that $X \in B^s_\tau(L_\tau(D))$ holds with probability one. Actually, $X \in B_2^{s+\delta}(L_\tau(D))$ holds with probability one if $\delta > 0$ is sufficiently small, and $B_2^{s+\delta}(L_\tau(D))$ is continuously embedded in $B^s_\tau(L_\tau(D))$. The moment bound from Theorem 2.6 therefore implies

$$\mathrm{E}\,\|X\|^2_{B^s_\tau(L_\tau(D))} \preceq \mathrm{E}\,\|X\|^2_{B_2^{s+\delta}(L_\tau(D))} < \infty,$$

which completes the proof. □

Remark 2.8. It follows from Corollary 2.7 that by choosing the sparsity parameter β close to one we get an *arbitrarily* high regularity in the nonlinear approximation scale of Besov spaces, provided that the wavelet basis is sufficiently smooth. This is obviously *not* possible in the classical L_2-Sobolev scale, see Theorem 2.6 with $p = q = 2$.

2.4. **Linear and Nonlinear Approximation.** In this section we study the approximation of X with respect to the L_2-norm. For linear approximation one considers the best approximation (i.e., orthogonal projection in our setting) from linear subspaces of dimension at most N. The corresponding *linear approximation error* of X is given by

$$e_N^{\mathrm{lin}}(X) = \inf\bigl(\mathrm{E}\,\|X - \widehat{X}\|^2_{L_2(D)}\bigr)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(D)$ such that

$$\dim(\mathrm{span}(\widehat{X}(\Omega))) \le N.$$

Theorem 2.9. *The linear approximation error satisfies*

$$e_N^{\rm lin}(X) \asymp (\ln N)^{\frac{\gamma d}{2}}\, N^{-\frac{\alpha+\beta-1}{2}}.$$

Proof. Truncating the expansion (3) of X we get a linear approximation

$$\widehat{X}_{j_1} = \sum_{j=j_0}^{j_1} \sum_{k\in\nabla_j} \sigma_j Y_{j,k} Z_{j,k}\, \psi_{j,k}, \tag{12}$$

which satisfies

$$\mathrm{E}\,\|X-\widehat{X}_{j_1}\|_{L_2(D)}^2 \asymp \sum_{j=j_1+1}^{\infty} \#\nabla_j \sigma_j^2 p_j \asymp \sum_{j=j_1+1}^{\infty} j^{\gamma d} 2^{-(\alpha+\beta-1)jd} \asymp j_1^{\gamma d} 2^{-(\alpha+\beta-1)j_1 d}. \tag{13}$$

Since $\dim(\mathrm{span}(\widehat{X}_{j_1}(\Omega))) \asymp \sum_{j=j_0}^{j_1} \#\nabla_j \asymp 2^{j_1 d}$, we get the upper bound as claimed.

In the proof of the lower bound we use the fact that $\psi_{j,k} = \Phi e_{j,k}$ for an orthonormal basis $(e_{j,k})_{j,k}$ in $L_2(D)$ and a bounded linear bijection $\Phi : L_2(D) \to L_2(D)$. This implies

$$e_N^{\rm lin}(X) \asymp e_N^{\rm lin}(\Phi^{-1}X).$$

Furthermore, $e_N^{\rm lin}(\Phi^{-1}X)$ depends on $\Phi^{-1}X$ only via its covariance operator $\widetilde{Q}$, which is given by

$$\widetilde{Q}\xi = \sum_{j=j_0}^{\infty} \sigma_j^2 p_j \sum_{k\in\nabla_j} \langle \xi, e_{j,k}\rangle_{L_2(D)}\, e_{j,k}, \tag{14}$$

cf. Remark 2.1. Consequently, the functions $e_{j,k}$ form an orthonormal basis of eigenfunctions of $\widetilde{Q}$ with associated eigenvalues $\sigma_j^2 p_j$. Due to a theorem by Micchelli and Wahba, see, e.g., [39, Prop. III.24], we get

$$e_N^{\rm lin}(\Phi^{-1}X) = \Bigg(\sum_{j=j_1+1}^{\infty} \#\nabla_j \sigma_j^2 p_j \Bigg)^{1/2}$$

if $N = \sum_{j=j_0}^{j_1} \#\nabla_j$. □

The best N-term (wavelet) approximation imposes a restriction only on the number

$$\eta(g) = \#\Big\{ \lambda\in\nabla : c_\lambda \neq 0, g = \sum_{\lambda\in\nabla} c_\lambda\, \psi_\lambda \Big\}$$

of non-zero wavelet coefficients of g. Hence the corresponding *error of best N-term approximation* for X is given by

$$e_N(X) = \inf\big(\mathrm{E}\,\|X-\widehat{X}\|_{L_2(D)}^2\big)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(D)$ such that

$$\eta(\widehat{X}(\omega)) \le N \quad P\text{-a.s.}$$

The analysis of $e_N(X)$ is based on the Besov regularity of X and the underlying wavelet basis Ψ. For deterministic functions x on D the error of best N-term approximation with respect to the L_2-norm is defined by

$$\sigma_N(x) = \inf\{\|x-\widehat{x}\|_{L_2(D)} : \widehat{x}\in L_2(D), \eta(\widehat{x}) \le N\}. \tag{15}$$

Clearly

$$e_N(X) = \big(\mathrm{E}(\sigma_N^2(X))\big)^{1/2}.$$

Theorem 2.10. *For every $\varepsilon > 0$, the error of best N-term approximation satisfies*

$$e_N(X) \preceq \begin{cases} N^{-1/\varepsilon}, & \text{if } \beta = 1 \\ N^{-\frac{\alpha+\beta-1}{2(1-\beta)}+\varepsilon}, & \text{otherwise.} \end{cases}$$

Proof. It suffices to consider the case $\beta < 1$. Let s and τ satisfy (11) with $t = 0$. Since $x \in B^s_\tau(L_\tau(D))$ implies

$$\sigma_N(x) \preceq \|x\|_{B^s_\tau(L_\tau(D))}\, N^{-s/d},$$

see [25] or [26] for this fundamental connection between Besov regularity and best N-term approximation, it remains to apply Corollary 2.7. □

For random functions it is also reasonable to impose a constraint on the average number of non-zero wavelet coefficients only, and to study the *error of best average N-term (wavelet) approximation*

$$e^{\mathrm{avg}}_N(X) = \inf\bigl(\mathrm{E}\,\|X - \widehat{X}\|^2_{L_2(D)}\bigr)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(D)$ such that

$$\mathrm{E}(\eta(\widehat{X})) \le N.$$

Theorem 2.11. *The error of best average N-term approximation satisfies*

$$e^{\mathrm{avg}}_N(X) \preceq \begin{cases} N^{\frac{\gamma d}{2}}\, 2^{-\frac{\alpha d N}{2}}, & \text{if } \beta = 1 \\ (\ln N)^{\frac{\gamma d}{2}}\, N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}, & \text{otherwise.} \end{cases}$$

Proof. Let $N_{j_1} = \mathrm{E}(\eta(\widehat{X}_{j_1}))$ for $\widehat{X}_{j_1}$ as in (12). Clearly

$$N_{j_1} = \sum_{j=j_0}^{j_1} \#\nabla_j p_j \asymp \sum_{j=j_0}^{j_1} 2^{(1-\beta)jd} \asymp \begin{cases} j_1, & \text{if } \beta = 1 \\ 2^{(1-\beta)j_1 d}, & \text{otherwise.} \end{cases}$$

In particular, $2^{j_1 d} \asymp N_{j_1}^{1/(1-\beta)}$ if $0 \le \beta < 1$. It remains to observe the error bound (13). □

The asymptotic behaviour of the linear approximation error $e^{\mathrm{lin}}_N(X)$ is determined by the *decay of the eigenvalues* $\sigma_j^2 p_j$ of the covariance operator $\widetilde{Q}$, see (14), i.e., it is essentially determined by the parameter $\alpha + \beta$. According to Theorem 2.6, the latter quantity also determines the regularity of X in the scale of Sobolev spaces $H^s(D)$.

For $0 < \beta \le 1$ nonlinear approximation is *superior* to linear approximation. More precisely, the following holds true. By definition, $e^{\mathrm{avg}}_N(X) \le e_N(X)$, and for $\beta > 0$ the convergence of $e_N(X)$ to zero is faster than that of $e^{\mathrm{lin}}_N(X)$. For $0 < \beta < 1$ the upper bounds for $e^{\mathrm{avg}}_N(X)$ and $e_N(X)$ slightly differ, and any dependence of $e_N(X)$ on the parameter γ is swallowed by the term N^ε in the upper bound. For linear and best average N-term approximation we have

$$e^{\mathrm{avg}}_{N^{1-\beta}}(X) \preceq e^{\mathrm{lin}}_N(X) \text{ if } 0 < \beta < 1$$

and

$$e^{\mathrm{avg}}_{c\ln N}(X) \preceq e^{\mathrm{lin}}_N(X) \text{ if } \beta = 1$$

with a suitably chosen constant $c > 0$.

Remark 2.12. We stress that for $0 < \beta \leq 1$ the *simulation* of the approximation $\widehat{X}_{j_1}$, which achieves the upper bound in Theorem 2.11, is possible at an *average computational cost* of order N_{j_1}. Let us briefly sketch the method of simulation. Put $n_j = \#\nabla_j$. For each level j we first simulate a binomial distribution with parameters n_j and p_j, which is possible at an average cost of order at most $n_j \cdot p_j$. Conditional on a realization $L(\omega)$ of this step, the locations of the non-zero coefficients on level j are uniformly distributed on the set of all subsets of $\{0, \dots, n_j\}$ of cardinality $L(\omega)$. Thus, in the second step, we employ acceptance-rejection to collect the elements of such a random subset sequentially. If $L(\omega) \leq n_j/2$, then all acceptance probabilities are at least $1/2$, and otherwise we switch to complements to obtain the same bound for the acceptance probability. In this way, the average cost of the second step is of order $n_j \cdot p_j$, too. In the last step we simulate the values of the non-zero coefficients. In total, the average computational cost for each level j is of order $n_j \cdot p_j$.

Remark 2.13. For Theorems 2.9 and 2.11 we only need the properties (A1) and (A2) of the basis Ψ, and (A2) enters only via the asymptotic behaviour of the parameters p_j and σ_j. After a lexicographic reordering of the indices (j,k) the two assumptions essentially amount to

$$X = \sum_{n=1}^{\infty} \sigma_n Y_n Z_n \psi_n$$

with any Riesz basis $(\psi_n)_{n\in\mathbb{N}}$ for $L_2(D)$, and

$$\sigma_n \asymp (\ln n)^{\gamma d/2} n^{-\alpha/2}$$

as well as independent random variables Y_n and Z_n, where Z_n is $N(0,1)$-distributed and Y_n is Bernoulli distributed with parameter

$$p_n \asymp n^{-\beta}.$$

Hence, Theorems 2.9 and 2.11 remain valid beyond the wavelet setting. For instance, let $p_n = 1$, which corresponds to $\beta = 0$ and $C_1 \geq 1$. Classical examples for Gaussian random functions on $D = [0,1]^d$ are the *Brownian sheet*, which corresponds to

$$\alpha = 2 \qquad \text{and} \qquad \gamma = 2(d-1)/d,$$

and *Lévy's Brownian motion*, which corresponds to

$$\alpha = (d+1)/d \qquad \text{and} \qquad \gamma = 0.$$

Theorem 2.9 is due to [37,54] for the Brownian sheet and due to [52] for Lévy's Brownian motion. See [39, Chap. VI] for further results and references on approximation of Gaussian random functions. For $\beta > 0$ our stochastic model provides sparse variants of general Gaussian random function.

3. Nonlinear Approximation for Elliptic Boundary Value Problems

We are interested in adaptive numerical wavelet algorithms for elliptic boundary value problems with random right-hand sides. As a particular, but nevertheless very important, model problem we are concerned with the Poisson equation on a bounded Lipschitz domain $D \subset \mathbb{R}^d$,

$$\begin{aligned} -\Delta U(\omega) &= X(\omega) \quad \text{in} \quad D, \\ U(\omega) &= 0 \quad \text{on} \quad \partial D, \end{aligned} \tag{16}$$

where the right-hand side $X : \Omega \to L_2(D) \subset H^{-1}(D)$ is a random function as described in Section 2.2. However, X is given as expansion in the dual basis $\tilde{\Psi}$.

As we will explain in the next section, the adaptive wavelet algorithms we want to employ are *optimal* in the sense that they asymptotically realize the approximation order of best N-term (wavelet) approximation. Such convergence analysis of adaptive algorithms relies on the energy norm, hence we study best N-term approximations of the random function $U : \Omega \to H^1(D)$ in $H^1(D)$. Analogously to (15) we introduce

$$\sigma_{N,H^1(D)}(u) = \inf\{\|u - \widehat{u}\|_{H^1(D)} : \widehat{u} \in H^1(D), \eta(\widehat{u}) \le N\}.$$

The quantity $\sigma_{N,H^1(D)}(U(\omega))$, where $U(\omega)$ is the exact solution of (16), serves as benchmark for the performance. To analyse the power of adaptive algorithms in the stochastic setting, we investigate the error

$$e_{N,H^1(D)}(U) = \inf(\mathrm{E}\,\|U - \widehat{U}\|^2_{H^1(D)})^{1/2},$$

with the infimum taken over all measurable mappings $\widehat{U} : \Omega \to H^1(D)$ such that

$$\eta(\widehat{X}(\omega)) \le N \quad P\text{-a.s.}$$

Clearly

$$e_{N,H^1(D)}(U) = \Big(\mathrm{E}(\sigma^2_{N,H^1(D)}(U))\Big)^{1/2}.$$

Theorem 3.1. *Suppose that $d \in \{2, 3\}$ and that the right-hand side X in (16) is of the form (3). Put*

$$\varrho = \min\left(\frac{1}{2(d-1)}, \frac{\alpha+\beta-1}{6} + \frac{2}{3d}\right).$$

Then, for every $\varepsilon > 0$, the error of the best N-term approximation satisfies

$$e_{N,H^1(D)}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. It is well known that for all $d \ge 1$

$$\sigma_{N,H^1(D)}(u) \preceq \|u\|_{B^r_\tau(L_\tau(D))} N^{-(r-1)/d}, \tag{17}$$

where

$$\frac{1}{\tau} = \frac{r-1}{d} + \frac{1}{2},$$

see, e.g., [15] for details. The next step is to control the norm of a solution u in the Besov space $B^r_\tau(L_\tau(D))$ in terms of the regularity of the right-hand side x of the Poisson equation. Let

$$-\frac{1}{2} < r^* < \frac{4-d}{2(d-1)},$$

and assume that $x \in H^{r^*}(D)$. We may apply the results from [15, 16] to conclude that $u \in B^{r-\delta}_\tau(L_\tau(D))$ for sufficiently small $\delta > 0$, where

$$r = \frac{r^*+5}{3}, \qquad \frac{1}{\tau} = \frac{r-\delta-1}{d} + \frac{1}{2}.$$

Moreover,

$$\|u\|_{B^{r-\delta}_\tau(L_\tau(D))} \preceq \|x\|_{B^{r^*}_2(L_2(D))}$$

and we can use (17), with r replaced by $r - \delta$, to derive

$$\sigma_{N,H^1(D)}(u) \preceq \|x\|_{B^{r^*}_2(L_2(D))} N^{-(r^*+2-3\delta)/(3d)}.$$

If, in addition, $r^*/d < (\alpha+\beta-1)/2$, then

$$e_{N,H^1(D)}(U) \preceq N^{-(r^*+2-3\delta)/(3d)}$$

follows from Theorem 2.6. □

Remark 3.2. In Theorem 3.1, we have concentrated on nonlinear approximations in $H^1(D)$, since this norm is the most important one from the numerical point of view, as we have briefly outlined above. In the deterministic setting, similar results for approximations in other norms, e.g., in L_2 or even weaker norms, also exist, see, e.g., [21] for details.

Remark 3.3. It is well known that the convergence order of classical nonadaptive *uniform* methods does not depend on the Besov regularity of the exact solution but on its Sobolev smoothness, see, e.g., [20, 21, 32] for details. However, on a Lipschitz domain, due to singularities at the boundary, the best one can expect is $U \in H^{3/2}(D)$, even for very smooth right-hand sides, see [31,33]. Therefore a uniform approximation scheme can only give the order $N^{-1/(2d)}$. In our setting, see Theorem 3.1, we have

$$\varrho > \frac{1}{2d},$$

so that for the problem (16) adaptive wavelet schemes are always superior when compared with uniform schemes.

Remark 3.4. With increasing values of α and β the smoothness of X increases, see Theorem 2.6. On a general Lipschitz domain, however, this does not necessarily increase the Besov regularity of the corresponding solution. This is reflected by the fact that the upper bound in Theorem 3.1 is at most of order $N^{-1/(2(d-1))}$.

For more specific domains, i.e., for polygonal domains, better results are available. Let D denote a simply connected polygonal domain in $\mathbb{R}^2$. Then, it is well known that if the right-hand side X in (16) is contained in $H^{r-1}(D)$ for some $r \geq 0$, the solution U can be uniquely decomposed into a regular part U_R and a singular part U_S, i.e., $U = U_R + U_S$, where $U_R \in H^{r+1}(D)$ and U_S belongs to a finite-dimensional space that only depends on the shape of the domain. This result was established by Grisvard, see [29], [30, Chapt. 4, 5], or [31, Sect. 2.7] for details.

Theorem 3.5. *Suppose that D is a simply connected polygonal domain in $\mathbb{R}^2$ and that the right-hand side X in* (16) *is of the form* (3). *Put*

$$\varrho = \frac{\alpha+\beta}{2}.$$

Then, for every $\varepsilon > 0$, error of the best N-term approximation satisfies

$$e_{N,H^1(D)}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. We apply the results from [29–31]. Let us denote the segments of ∂D by $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_M$ with open sets Γ_ℓ numbered in positive orientation. Furthermore, let Υ_ℓ denote the endpoint of Γ_ℓ and let χ_ℓ denote the measure of the interior angle at Υ_ℓ. We introduce polar coordinates $(\kappa_\ell, \theta_\ell)$ in the vicinity of each vertex Υ_ℓ, and for $n \in \mathbb{N}$ and $\ell = 1, \ldots, M$ we introduce the functions

$$\mathcal{S}_{\ell,n}(\kappa_\ell, \theta_\ell) = \zeta_\ell(\kappa_\ell) \cdot \kappa_\ell^{\lambda_{\ell,n}} \sin(n\pi\theta_\ell/\chi_\ell),$$

when $\lambda_{\ell,n} = n\pi/\chi_\ell$ is not an integer, and

$$\mathcal{S}_{\ell,n}(\kappa_\ell, \theta_\ell) = \zeta_\ell(\kappa_\ell) \cdot \kappa_\ell^{\lambda_{\ell,n}} [\log \kappa_\ell \sin(n\pi\theta_\ell/\chi_\ell) + \theta_\ell \cos(n\pi\theta_\ell/\chi_\ell)]$$

otherwise. Here ζ_ℓ denotes a suitable C^∞ truncation function.

Consider the solution $u = u_R + u_S$ of the Poisson equation with the right-hand side $x \in H^{r-1}(D)$, and assume that

$$r \notin \{\lambda_{\ell,n} : n \in \mathbb{N},\ \ell = 1, \dots, M\}.$$

Then one has $u_R \in H^{r+1}(D)$ and $u_S \in \mathcal{N}$ for

$$\mathcal{N} = \operatorname{span}\big\{\mathcal{S}_{\ell,n} : 0 < \lambda_{n,l} < r\big\}.$$

We have to estimate the Besov regularity of both, u_S and u_R, in the scale given by (11) with $t = 1$, i.e,

$$\frac{1}{\tau} = \frac{s}{2}.$$

Classical embeddings of Besov spaces imply that $u_R \in B^s_\tau(L_\tau(D))$ for every $s < r + 1$. Moreover, it has been shown in [14] that $\mathcal{N} \subset B^s_\tau(L_\tau(D))$ for every $s > 0$. We conclude that $u \in B^s_\tau(L_\tau(D))$ for every $s < r + 1$.

To estimate u, we argue as follows. Let γ_ℓ be the trace operator with respect to the segment Γ_ℓ. Grisvard has shown that the Laplacian Δ maps the direct sum

$$H = \big\{u \in H^{r+1}(D) : \gamma_\ell u = 0, \ell = 1, \dots, M\big\} + \mathcal{N}$$

onto $H^{r-1}(D)$, cf. [30, Thm. 5.1.3.5]. Note that $(H, \|\cdot\|_H)$ is a Banach space where

$$\|u\|_H = \|u_R\|_{H^{r+1}(D)} + \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<r} |c_{\ell,n}|$$

for

$$u_S = \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<r} c_{\ell,n}\, \mathcal{S}_{\ell,n}.$$

It has been shown in [19] that the solution operator Δ^{-1} is continuous as a mapping from $H^{r-1}(D)$ onto H. Therefore

$$\|u\|_{B^s_\tau(L_\tau(D))} \preceq \|u_R\|_{H^{r+1}(D)} + \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<t} |c_{\ell,n}| = \|u\|_H \preceq \|x\|_{H^{r-1}(D)}$$

for every $s < r + 1$.

Finally, by Theorem 2.6, $X \in H^{r-1}(D)$ with probability one and $\mathrm{E}\,\|X\|^2_{H^{r-1}(D)} < \infty$ if $r < \alpha + \beta$. Now the upper bound for $e_{N,H^1(D)}(U)$ follows by proceeding as in the proof of Theorem 3.1. □

Let us briefly discuss the case of a C^∞-domain D. In this case, the problem is completely regular and no singularities induced by the shape of the domain can occur. However, similar to Corollary 2.7, it is a remarkable fact that for β close to one an *arbitrarily* high order of convergence can be realized.

Theorem 3.6. *Suppose that D is a bounded C^∞-domain in $\mathbb{R}^d$ and that the right-hand side X in (16) is of the form (3). Moreover, assume that $\beta < 1$ and put*

$$\varrho = \frac{1}{1-\beta}\left(\frac{\alpha-1}{2}+\beta\right)+\frac{1}{d}.$$

Then, for every $\varepsilon > 0$, the error of best N-term approximation satisfies

$$e_{N,H^1(D)}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. In order to ensure that the condition $s > d(1/\tau - 1)_+$ is satisfied, let us consider the scale

$$\frac{1}{\tau} = \frac{s}{d} + 1 - \delta$$

for some arbitrarily small parameter $\delta > 0$. Given α and β, a similar calculation as in the proof of Corollary 2.7 shows that $X \in B^s_\tau(L_\tau(D))$ holds with probability one and $\mathrm{E}\,\|X\|^2_{B^s_\tau(L_\tau(D))} < \infty$ if

$$s < \frac{d}{1-\beta}\left(\frac{\alpha-1}{2}+\beta(1-\delta)\right).$$

Since the problem is regular, the solution u of the Poisson equation with right-hand side $x \in B^s_\tau(L_\tau(D))$ satisfies $u \in B^{s+2}_\tau(L_\tau(D))$ with

$$\|u\|_{B^{s+2}_\tau(L_\tau(D))} \preceq \|x\|_{B^s_\tau(L_\tau(D))},$$

see [41, Chap. 3] or [48, Thm. 4.3]. By a classical embedding of Besov spaces we obtain

$$\|u\|_{B^{s+2}_{\tau^*}(L_{\tau^*}(D))} \preceq \|u\|_{B^{s+2}_\tau(L_\tau(D))}$$

for the nonlinear approximation scale

$$\frac{1}{\tau^*} = \frac{(s+2)-1}{d} + \frac{1}{2}.$$

An application of (17) yields

$$\sigma_{N,H^1(D)}(u) \preceq \|u\|_{B^{s+2}_{\tau^*}(L_{\tau^*}(D))} N^{-(s+1)/d}.$$

We conclude that

$$e_{N,H^1(D)}(U) \preceq N^{-(s+1)/d}.$$

Let δ tend to zero to obtain the result as claimed. □

For $\beta = 1$ the estimate from Theorem 3.6 is valid for arbitrarily large ϱ, provided that the primal and dual wavelet bases are sufficiently smooth, cf. assumption (A5).

4. Adaptive Wavelet Methods for Elliptic Operator Equations

In this section, let us briefly review the class of deterministic adaptive wavelet algorithms that will be applied in the numerical discretisation of (16).

4.1. **Elliptic PDEs in Wavelet Coordinates.** The stochastic boundary value problem under consideration can be written as an elliptic random operator equation

$$AU(\omega) = X(\omega), \tag{18}$$

where $\omega \in \Omega$ and $A : H \to H'$ is a linear, boundedly invertible mapping between a separable Hilbert space H and its normed dual. The associated bilinear form $a(u,v) = \langle Au, v\rangle_{H'\times H}$ is bounded, $|a(u,v)| \preceq \|u\|_H\|v\|_H$, and elliptic, $a(u,u) \succeq \|u\|_H^2$. In case that $H = H_0^1(D)$, as in (16), and whenever we have a wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ of $L_2(D)$ fulfilling the assumptions (A1)–(A5), the equation (18) has an equivalent reformulation

$$\mathbf{AU}(\omega) = \mathbf{X}(\omega) \tag{19}$$

in wavelet coordinates, where we define $\mathbf{A} = (\mathbf{A}_{\lambda,\mu})$ and $\mathbf{X} = (\mathbf{X}_\lambda)$ by

$$\mathbf{A}_{\lambda,\mu} = 2^{-(|\mu|+|\lambda|)}a(\psi_\mu, \psi_\lambda)$$

and

$$\mathbf{X}_\lambda(\omega) = 2^{-|\lambda|}\big\langle X(\omega), \psi_\lambda\big\rangle_{H^{-1}(D)\times H_0^1(D)}$$

for all $\mu, \lambda \in \nabla$. Here equivalence means the following. For each solution $\mathbf{U}(\omega) \in \ell_2(\nabla)$ to (19), the associated wavelet expansion

$$U(\omega) = \sum_{\lambda\in\nabla} 2^{-|\lambda|}\mathbf{U}_\lambda(\omega)\psi_\lambda \in H$$

solves (18). Conversely, the unique expansion coefficients

$$\mathbf{U}(\omega)_\lambda = 2^{|\lambda|}\langle\tilde{\psi}_\lambda, U(\omega)\rangle_{H^{-1}(D)\times H_0^1(D)}$$

of a solution $U(\omega) \in H$ to (18) satisfy (19).

Such an equivalence still holds if the underlying wavelet basis is replaced by a wavelet frame, i.e., a redundant wavelet system which allows for stable analysis and synthesis operations. On general domains, e.g., polygonal domains in $\mathbb{R}^2$, wavelet frames are much easier to construct than bases, see [18, 44]. We tacitly assume that the chosen wavelet frame fulfils assumptions (A2)–(A5), see [17] for details on the characterization of Besov spaces. In the case of frames, each solution $\mathbf{U}(\omega) \in \ell_2(\nabla)$ to (19) still expands into a solution $U(\omega) \in H$ to (18). However, uniqueness of the expansion coefficients $\mathbf{U}(\omega)$ of $U(\omega)$ can only be expected to hold within the range of $\mathbf{A}$, being a proper subset of $\ell_2(\nabla)$.

4.2. **Adaptive Wavelet Frame Methods.** In the numerical experiments, we will employ a class of adaptive wavelet methods that also works with frames. Let $\Psi = \bigcup_{i=1}^M \Psi_i$ be a finite union of wavelet bases Ψ_i for $L_2(D_i)$, subordinate to an overlapping partition of D into patches D_i. The properties (A1)–(A5) are assumed to hold for each basis Ψ_i in a patchwise sense. In that case, the main diagonal blocks $\mathbf{A}_{i,i}$ of $\mathbf{A}$ are invertible, so that the following abstract Gauss-Seidel iteration

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + \mathbf{R}(\mathbf{X}(\omega) - \mathbf{AU}^{(n)}), \qquad n = 0, 1, \dots \tag{20}$$

with

$$\mathbf{R} = (\mathbf{A}_{j,j})_{j\le i}^{-1}$$

makes sense. The convergence properties of the iteration (20) and, in particular, of fully adaptive variants involving inexact operator evaluations and inexact applications of the preconditioner $\mathbf{R}$ have been recently analysed, see [46, 53]. In order to turn the abstract iteration (20) into an implementable scheme, all infinite-dimensional quantities have to

be replaced by computable ones. Such realizations involve the inexact evaluation of the right-hand side $\mathbf{X}(\omega)$ and of the various biinfinite matrix-vector products $\mathbf{A}_{i,j}\mathbf{v}$, both enabled by the compression properties of the wavelet system Ψ. We refer to [6, 7, 44] for details and properties of the corresponding numerical subroutines.

A full convergence and cost analysis of the resulting wavelet frame domain decomposition algorithm is available in case that a particular set of quadrature rules is used in the overlapping regions of the domain decomposition, see [45], and under the assumption that the local subproblems $\mathbf{A}_{i,i}\mathbf{v} = \mathbf{g}$ are solved with a suitable adaptive wavelet scheme, e.g., the wavelet-Galerkin methods from [7, 28]. From the findings of [45, 46] and by the properties of the aforementioned numerical subroutines, an implementable numerical routine $\mathbf{SOLVE}[\omega, \varepsilon] \to \mathbf{U}_\varepsilon(\omega)$ with the following key properties exists:

- For each $\varepsilon > 0$ and $\omega \in \Omega$, **SOLVE** outputs a finitely supported sequence $\mathbf{U}_\varepsilon(\omega)$ with guaranteed accuracy
$$\|\mathbf{U}(\omega) - \mathbf{U}_\varepsilon(\omega)\|_{\ell_2(\nabla)} \leq \varepsilon \ (\textit{convergence});$$
- Whenever the best N-term approximation of $U(\omega)$ with respect to Ψ converges at a rate $s > 0$ in $H^1(D)$, i.e.,
$$\big\|U(\omega)\big\|_{\mathcal{A}^s(H^1)} := \sup_{N\in\mathbb{N}_0} N^s \sigma_{N,H^1}\big(U(\omega)\big) < \infty,$$
then the outputs $\mathbf{U}_\varepsilon(\omega)$ realize the same work/accuracy ratio, as $\varepsilon \to 0$, i.e.,
$$\#\operatorname{supp}\mathbf{U}_\varepsilon(\omega) \preceq \varepsilon^{-1/s}\big\|U(\omega)\big\|^{1/s}_{\mathcal{A}^s(H^1)} \ (\textit{convergence rates});$$
- The associated computational work asymptotically scales in the same way,
$$\#\operatorname{flops}_\varepsilon(\omega) \preceq \varepsilon^{-1/s}\big\|U(\omega)\big\|^{1/s}_{\mathcal{A}^s(H^1)} \ (\textit{linear cost}).$$

Generalizations of these properties towards the average case setting are straightforward. In fact, the results from Sections 2 and 3 provide upper bounds for $\big\|U(\omega)\big\|_{\mathcal{A}^s(H^1)}$ for certain values of s in terms of suitable Besov norms of the right-hand side $X(\omega)$, and the latter norms may be chosen to have arbitrarily high moments.

It is unclear whether these properties of **SOLVE** simultaneously hold for approximations of $U(\omega)$ in weaker norms than $H^1(D)$. To our best knowledge, a counterpart of Nitsche duality arguments is so far unknown in the setting of adaptive wavelet approximation. However, many numerical experiments on adaptive wavelet methods indicate that higher convergence rates in weaker norms can indeed be expected.

5. Numerical Experiments

In this section we present numerical experiments to complement our analysis and to demonstrate its applicability. First, we consider the random functions as introduced in Section 2, and thereafter, we study the adaptive numerical treatment of elliptic stochastic equations as outlined in the Sections 3 and 4. Throughout this section we focus on the impact of the parameters α and β of the stochastic model. So, we set $\gamma = 0$ and, in order to simplify the presentation, we only consider the case $\beta < 1$.

On the domain $D = [0, 1]$ the numerical experiments were performed by using a biorthogonal spline wavelet basis as constructed in [38]. The primal wavelets consist of cardinal splines of order $m = 3$, i.e., they are piecewise quadratic polynomials, and the

condition (A4) is satisfied with $\tilde{m} = 5$. The wavelet basis satisfies (A5) along the nonlinear approximation scale with $s_1 = 3$, while $s_1 = 2.5$ along the linear approximation scale, see [25]. Moreover, $j_0 = 2$ and $\#\nabla_{j_0} = 10$, while $\#\nabla_j = 2^j$ for $j > j_0$.

In the stochastic model (3) for X we set

$$C_1 = 2^{\beta j_0}, \qquad C_2 = 2^{\alpha j_0},$$

which means that sparsity is only induced at levels $j > j_0$ and the coefficients at level j_0 are standard normally distributed. This ensures that we keep the entire polynomial part of X.

5.1. **Realizations and Moments of Besov Norms of X.** First, let us illustrate the impact of the parameters α and β on individual realizations of X as well as on the sparsity and decay of its wavelet coefficients in the case $D = [0, 1]$. The parameter α influences decay, while β induces sparsity patterns in the wavelet coefficients.

In any simulation, only a finite number of coefficients can be handled. Therefore, we truncate the wavelet expansion in a suitable way, i.e., $\widehat{X}_{j_1}$ is the truncation of X at level j_1, see (12). Theorem 2.11 provides an error bound for the approximation of X by $\widehat{X}_{j_1}$ in terms of the expected number of non-zero coefficients. An efficient way of simulating $\widehat{X}_{j_1}$ is presented in Remark 2.12. Specifically, we choose

$$\alpha \in \{2.0, 1.8, 1.5, 1.2\}$$

and

$$\beta = 2 - \alpha,$$

which is motivated as follows. At first, $\alpha = 2$ and $\beta = 0$ corresponds to the smoothness of a Brownian motion, see Remark 2.13, and secondly, according to Theorems 2.9 and 2.11 for our choice of α and β the order of linear approximation is kept constant while the order of best average N-term approximation increases with β. The two underlying scales of Besov spaces $B^s_\tau(L_\tau(D))$ are the linear approximation scale, where $\tau = 2$, and the nonlinear approximation scale, where $1/\tau = s + 1/2$, see (11) with $t = 0$ for L_2-approximation.

We set

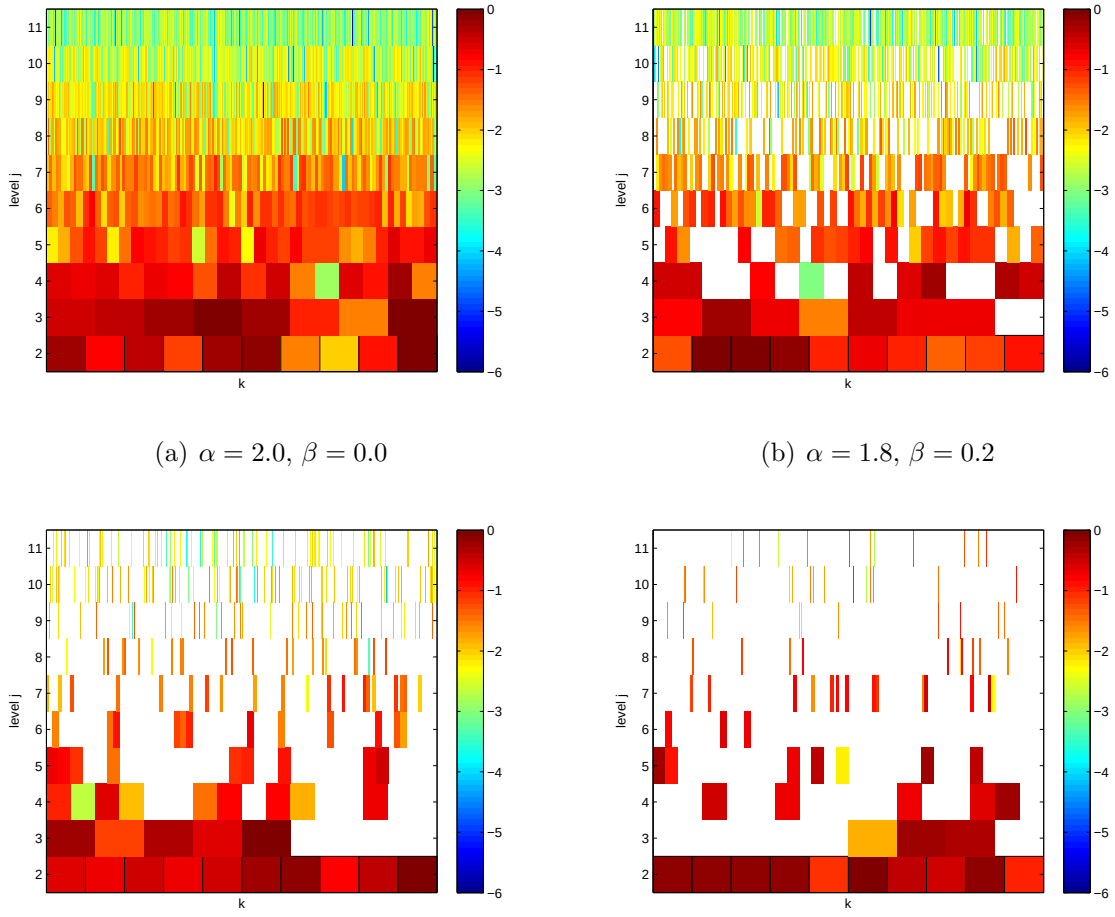
$$V_{j_1} = \max_{j \le j_1} \max_{k \in \nabla_j} \sigma_j Y_{j,k} |Z_{j,k}|$$

in order to normalize the absolute values of the coefficients. Figure 1 shows realizations of the normalized absolute values $\sigma_j Y_{j,k}|Z_{j,k}|/V_{j_1}$ of all coefficients up to level

$$j_1 = 11.$$

It exhibits that the parameter β induces sparsity patterns, for larger values of β more coefficients are zero and the wavelet expansion of X is sparser. Figure 2 illustrates the corresponding sample functions. We observe that for $\beta = 0$ the sample function is irregular everywhere, and by increasing β the irregularities become more and more isolated. This does not affect the L_2-Sobolev smoothness, while on the other hand it is well known that piecewise smooth functions with isolated singularities have a higher Besov smoothness on the nonlinear approximation scale. According to Theorem 2.6 and Corollary 2.7, X belongs to a space on the linear approximation scale with probability one if and only if

$$s < \frac{1}{2}, \tag{21}$$



(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 1.8$, $\beta = 0.2$

(c) $\alpha = 1.5$, $\beta = 0.5$

(d) $\alpha = 1.2$, $\beta = 0.8$

FIGURE 1. Absolute values of normalized coefficients of $\widehat{X}_{j_1}(\omega)$

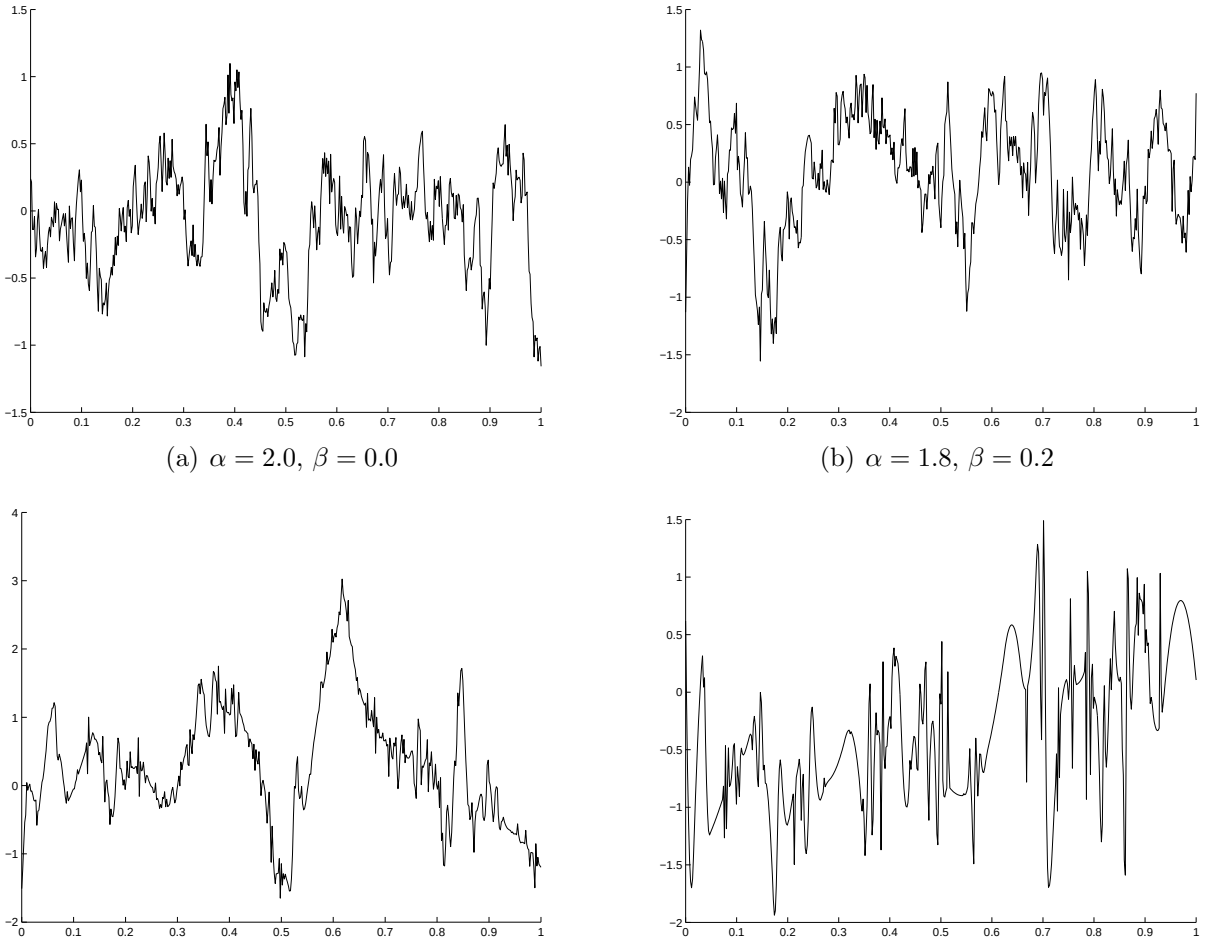
while X belongs to a space on the nonlinear approximation scale with probability one if and only if

$$s < \frac{1}{2(1-\beta)}. \tag{22}$$

Hence, these upper bounds reflect the orders of convergence for linear and best average N-term approximation, respectively.

Moments of Besov norms or of equivalent norms on sequence spaces appear in unspecified constants in the error bounds that are derived in Sections 2 and 3. Here we consider $D = [0,1]$, and the moments of X along the linear and nonlinear approximation scale. Put $b_j(s,\tau) = 2^{j(s+(\frac{1}{2}-\frac{1}{\tau}))\tau}$. By (A5) we get

$$\|X\|^{\tau}_{B^s_\tau(L_\tau(D))} \asymp \sum_{j=j_0}^{\infty} b_j(s,\tau)\, \sigma_j^{\tau} \sum_{k\in\nabla_j} Y_{j,k}|Z_{j,k}|^{\tau}.$$



(a) $\alpha = 2.0$, $\beta = 0.0$ (b) $\alpha = 1.8$, $\beta = 0.2$

(c) $\alpha = 1.5$, $\beta = 0.5$ (d) $\alpha = 1.2$, $\beta = 0.8$

FIGURE 2. Sample function of $\widehat{X}_{j_1}(\omega)$

Denoting with ν_τ the absolute moment of order τ of the standard normal distribution, i.e., $\nu_\tau = 2^{\tau/2}\,\Gamma((\tau+1)/2)/\pi^{1/2}$, we obtain $\mathrm{E}\,\|X\|^\tau_{B^s_\tau(L_\tau(D))} \asymp M(s,\tau)$ with

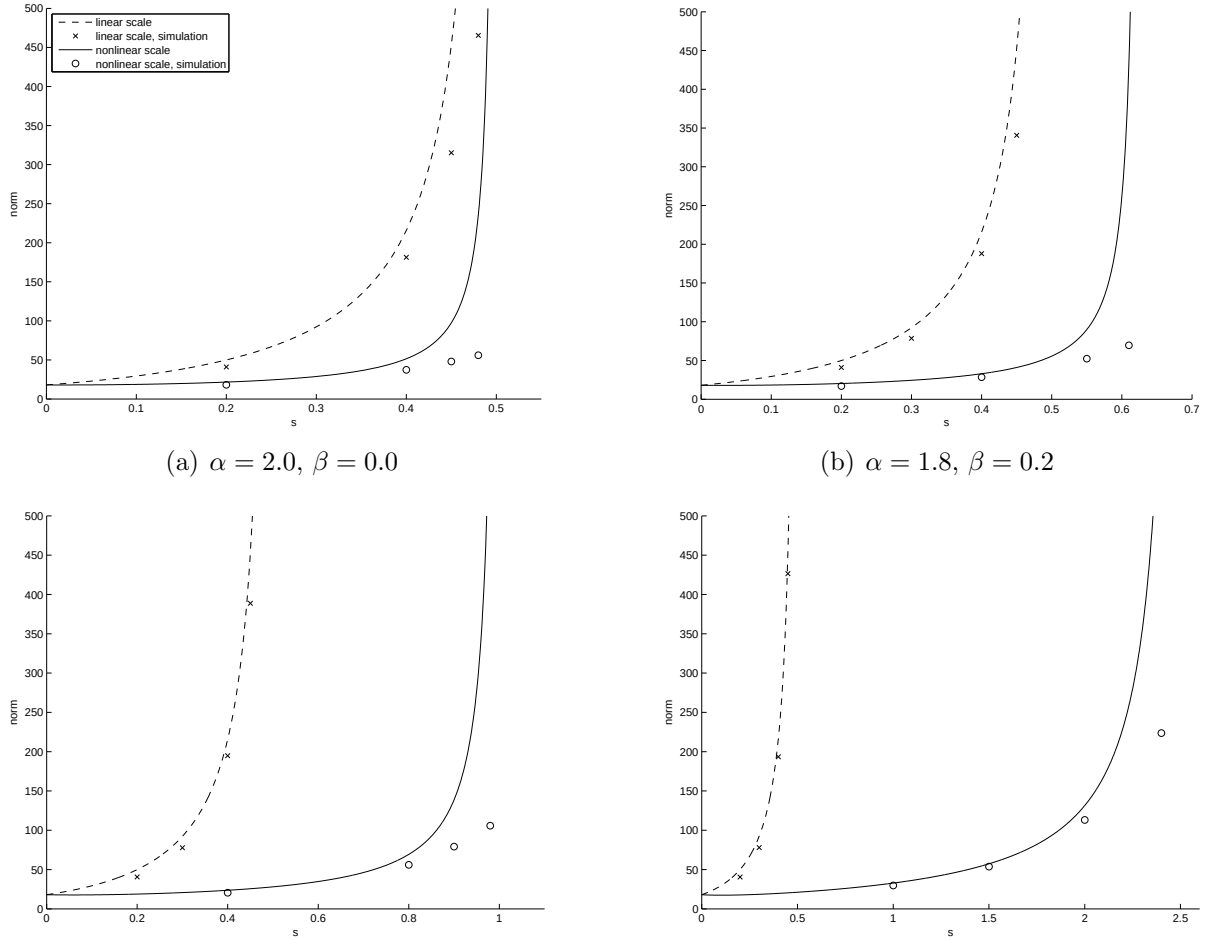
$$M(s,\tau) = \nu_\tau \sum_{j=j_0}^{\infty} b_j(s,\tau)\,\#\nabla_j \sigma_j^\tau p_j.$$

Figure 3 contains the graphs of M along the linear and nonlinear approximation scales, i.e., $s \mapsto M(s,2)$ and $s \mapsto M(s,1/(s+1/2))$, for the selected values of α and β. Note that the upper bounds (21) and (22) also provide the location of the singularities of M along the two scales.

The effect of truncation and finite sample size is illustrated in Figure 3, too, by presenting sample means of the right-hand side in

$$\|\widehat{X}_{j_1}\|^\tau_{B^s_\tau(L_\tau(D))} \asymp \sum_{j=j_0}^{j_1} b_j(s,\tau)\cdot\sigma_j^\tau \sum_{k\in\nabla_j} Y_{j,k}|Z_{j,k}|^\tau.$$

Specifically, for each scale and each choice of α and β we consider 4 different values of s and use 1000 independent samples. Moreover, for each choice of the parameters α and β, the truncation level j_1 is chosen according to Table 1 so that the expected number



(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 1.8$, $\beta = 0.2$

(c) $\alpha = 1.5$, $\beta = 0.5$

(d) $\alpha = 1.2$, $\beta = 0.8$

FIGURE 3. $\mathrm{E}\,\|X\|^{\tau}_{B^s_\tau(L_\tau(D))}$ along the linear and nonlinear approximation scales

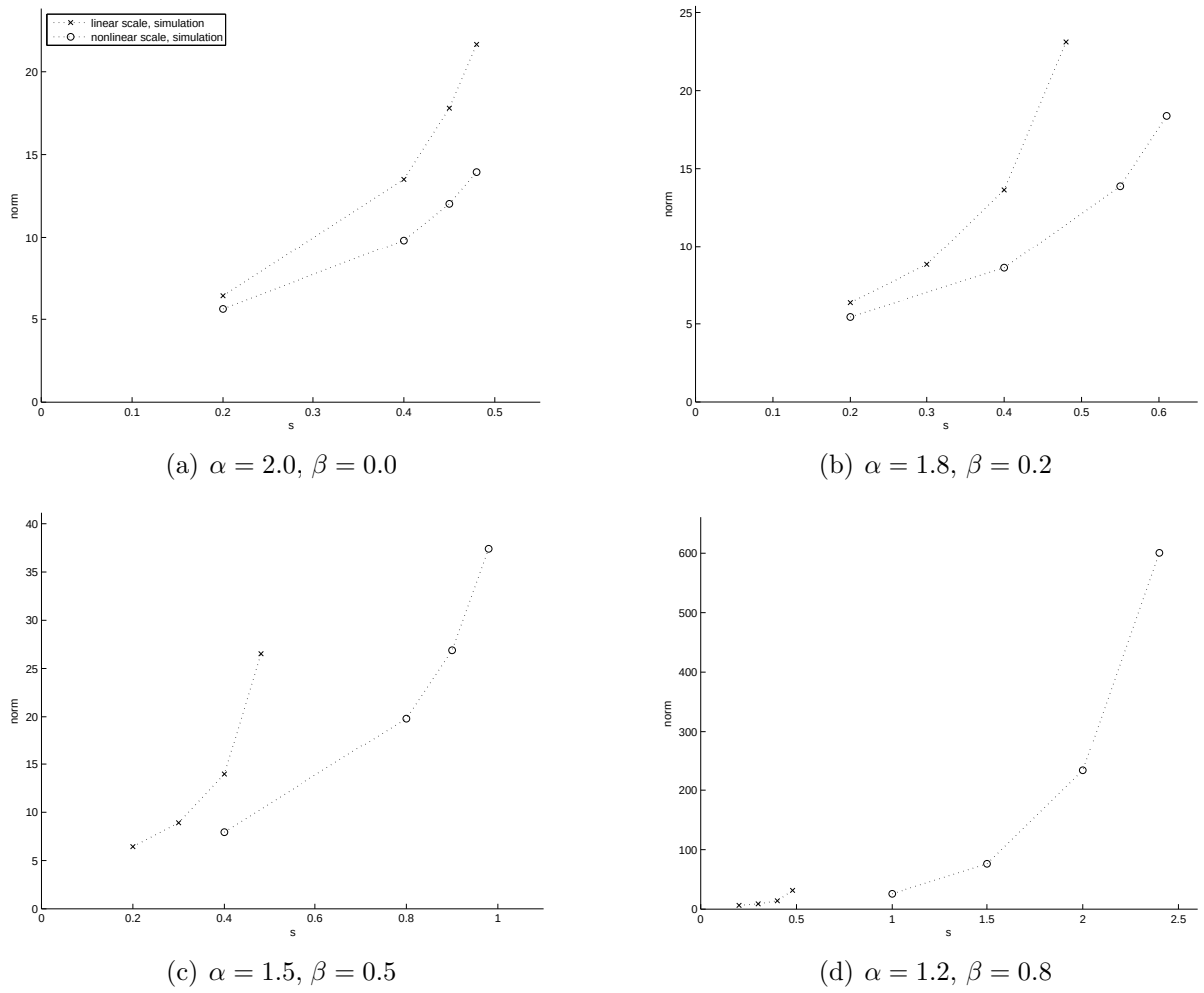
α	2.0	1.8	1.5	1.2
j_1	20	24	36	78

TABLE 1. Truncation levels in Figures 3 and 4

$\sum_{j=j_0}^{j_1} \#\nabla_j \cdot p_j$ of non-zero coefficients is approximately 10^6 in all cases. We observe the strongest impact of truncation for the nonlinear approximation scales and small values of β. We add that confidence intervals for the level 0.95 are of length less than 10 in all cases.

Likewise we proceed for $(\mathrm{E}\,\|X\|^2_{B^s_\tau(L_\tau(D))})^{1/2}$ along the linear and nonlinear approximation scales. These quantities appear as unspecified constants in the error bounds of Sections 2 and 3, see, e.g., (17). Figure 4 shows the results after truncation, sampling, and applying the norm equivalence. It is worth noting that the sampled Besov norms are smaller than the sampled L_2-Sobolev norms. We add that confidence intervals for the level 0.95 are of length less than one percent of the estimate in all cases.

5.2. **Adaptive Numerical Schemes.** Here we illustrate the impact of α and β on approximation rates of elliptic stochastic equations as outlined in Sections 3 and 4. In the



(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 1.8$, $\beta = 0.2$

(c) $\alpha = 1.5$, $\beta = 0.5$

(d) $\alpha = 1.2$, $\beta = 0.8$

FIGURE 4. $(\mathrm{E}\,\|X\|^2_{B^s_\tau(L_\tau(D))})^{1/2}$ along the linear and nonlinear approximation scales

one-dimensional case the equation is given by

$$\begin{aligned} -U''(\cdot,\omega) &= X(\cdot,\omega), \\ U(0,\omega) &= U(1,\omega) = 0 \end{aligned}$$

on $D = [0,1]$.

The aim is to investigate if for this model problem adaptive wavelet algorithms are superior when compared with uniform schemes. As we know the order of convergence of uniform methods is determined by the L_2-Sobolev regularity of the exact solution whereas the order of convergence of adaptive schemes is determined by the Besov smoothness, see (17) and Remark 3.3.

We therefore study an example where the Sobolev smoothness of the right-hand side X stays fixed while the Besov regularity changes. This is achieved by choosing parameter values α and β such that the sum $\alpha + \beta$, which determines the L_2-Sobolev regularity, is kept constant, cf. Section 5.1. Then, letting β tend to one increases the Besov smoothness significantly, see Corollary 2.7. Since the problem is completely regular, these interrelations immediately carry over to the exact solution, see [48, Theorem 4.3].

The numerical experiment is carried out and evaluated as follows. On input $\delta > 0$ the adaptive wavelet scheme computes an N-term approximation $\widehat{U}(\cdot,\omega)$ to $U(\cdot,\omega)$, whose error with respect to the H^1-norm is at most δ. The number N of terms depends on δ as well as on ω via the right-hand side $X(\cdot,\omega)$, and only a finite number of wavelet coefficients of $X(\cdot,\omega)$ are used to compute $\widehat{U}(\cdot,\omega)$. We determine $U(\cdot,\omega)$ in a master computation with a very high accuracy and then use the norm equivalence (A5) for the space $H^1(D)$. The master computation employs a uniform approximation with truncation level $j_1 = 11$ for the right-hand side. To get a reliable estimate for the average number $\mathrm{E}(\eta(\widehat{U}))$ of non-zero wavelet coefficients of $\widehat{U}$ and for the error $(\mathrm{E}\,\|U-\widehat{U}\|^2_{H^1(D)})^{1/2}$ we use 1000 independent samples of truncated right-hand sides. This procedure is carried out for 18 different values of δ; the results are presented together with a regression line, whose slope yields an estimate for the order of convergence. For the uniform approximation we proceed in the same way. The only difference is that, instead of δ, a fixed truncation level for the approximation of the left-hand side is used, and therefore no estimate is needed for the number of non-zero coefficients. As for the adaptive scheme we use 1000 independent samples for six different truncation levels, $j = 4,\dots,9$. We add that confidence intervals for the level 0.95 are of length less than three percent of the estimate in all cases.

In the first experiment we choose

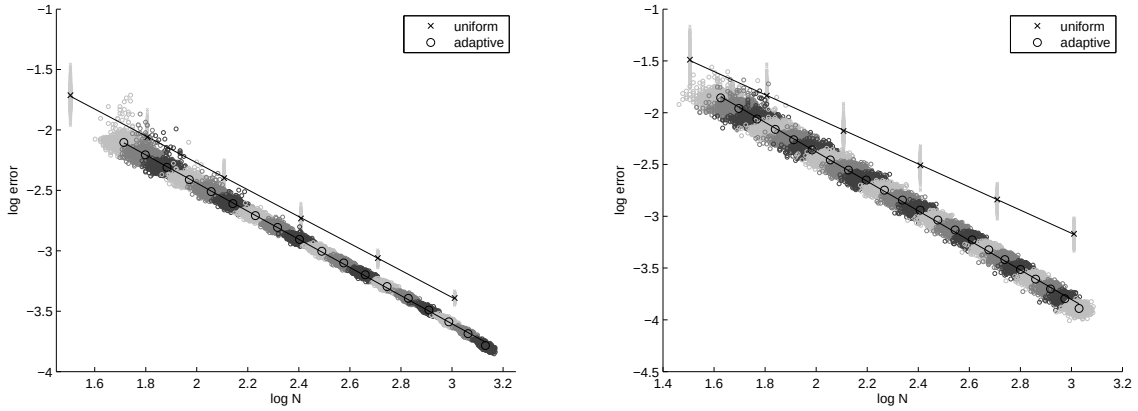
$$\alpha = 0.9, \qquad \beta = 0.2,$$

i.e., the right-hand side is contained in $H^s(D)$ only for $s < 0.05$. Consequently, the solution is contained in $H^s(D)$ with $s < 2.05$. An optimal uniform approximation scheme with respect to the H^1-norm yields the approximation order $1.05 - \varepsilon$ for every $\varepsilon > 0$. This is observed in Figure 5(a), where the empirical order of convergence for the uniform approximation is 1.113. For the relatively small value of $\beta = 0.2$, the Besov smoothness, and therefore the order of best N-term approximation, is not much higher. In fact, by inserting the parameters into Theorem 3.6 with $d = 1$, we get the approximation order $\varrho - \varepsilon$ with $\varrho = 19/16 = 1.1875$. This is also reflected in Figure 5(a), where the empirical order of convergence for the adaptive wavelet scheme is 1.164. In both cases the numerical results match very well the asymptotic error analysis, and both methods exhibit almost the same order of convergence. Nevertheless, even in this case adaptivity slightly pays off for the same regularity parameter, since the Besov norm is smaller than the Sobolev norm, which yields smaller constants.

The picture changes for higher values of β. As a second test case, we choose

$$\alpha = 0.4, \qquad \beta = 0.7.$$

Then, the Besov regularity is considerably higher. In fact, from Theorem 3.6 with $d = 1$ we would expect the convergence rate $\varrho - \varepsilon$ with $\varrho = 7/3$, provided that the wavelet basis indeed characterizes the corresponding Besov spaces. It is well known that a tensor product spline wavelet basis of order m in dimension d has this very property for $B^s_\tau(L_\tau)$ with $1/\tau = s - 1/2$ and $s < s_1 = m/d$, see [6, Theorem 3.7.7]. In our case, $s_1 = 3$, so $\varrho = 2$ is the best we can expect. From Figure 5(b), we observe that the empirical order of convergence is slightly lower, namely 1.425. The reason is that the Besov smoothness of the solution is only induced by the right-hand side which, in a Galerkin approach, is expanded in the *dual* wavelet basis. Estimating the Hölder regularity of the dual wavelet basis $\tilde{\Psi}$, see [50], it turns out that this wavelet basis is only contained in $W^s(L_\infty)$ for $s < 0.55$. Therefore, by using classical embeddings of Besov spaces, it is only ensured that this



(a) $\alpha = 0.9$, $\beta = 0.2$ (b) $\alpha = 0.4$, $\beta = 0.7$

FIGURE 5. Error and (expected) number of non-zero coefficients

wavelet basis characterizes Besov spaces $B^s_\tau(L_\tau)$, with the same smoothness parameter. Consequently, the solution U is only contained in the spaces $B^s_\tau(L_\tau)$ with $1/\tau = s - 1/2$ and $s < 2.55$ which gives an approximation order $\varrho - \varepsilon$ with $\varrho = 1.55$. This is matched very well in Figure 5(b). For uniform approximation the empirical order of convergence is 1.115 and thus does not differ from the result in the first experiment.

Our sample problem in the two-dimensional case is the Poisson equation (16) with zero-Dirichlet boundary conditions on the L-shaped domain

$$D = (-1, 1)^2 \setminus [0, 1)^2.$$

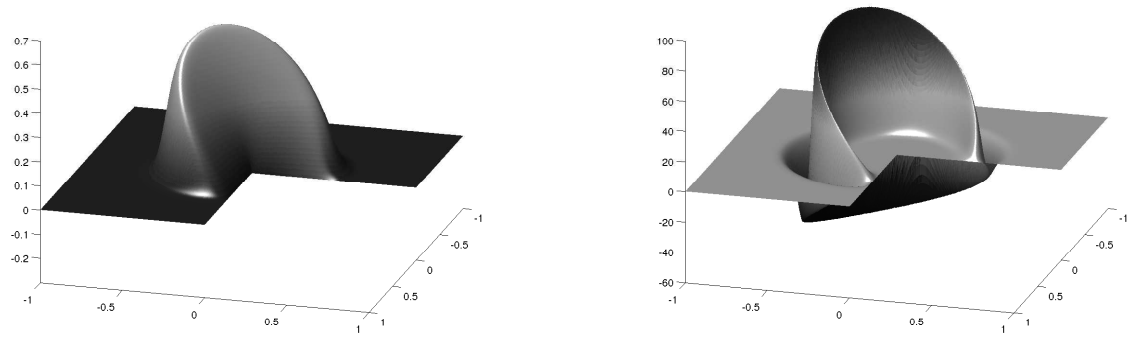
Here, as outlined in Section 4, we are going to apply domain decomposition methods based on wavelet frames. We consider the right-hand side $X = x + \tilde{X}$, where x is a known deterministic function and $\tilde{X}$ is generated by the stochastic model from Section 2.2, but based on frame decompositions. This means that we add a noise term to a deterministic right-hand side. Specifically, we consider perturbed versions of the well-known equation that is defined by the exact solution

$$u(r, \theta) = \zeta(r) r^{2/3} \sin\Big(\frac{2}{3}\theta\Big),$$

where (r, θ) are polar coordinates with respect to the re-entrant corner, see Figure 6. Then, u is one of the singularity functions as introduced in the proof of Theorem 3.5. It has a relatively low Sobolev regularity while its Besov smoothness is arbitrary high, see again the proof of Theorem 3.5 for details. For functions of this type we expect that adaptivity pays off. In Figure 7 we show two solutions to realisations of X for the parameter combination $\alpha = 1$ together with $\beta = 0.1$ and its sparse counterpart $\beta = 0.9$.

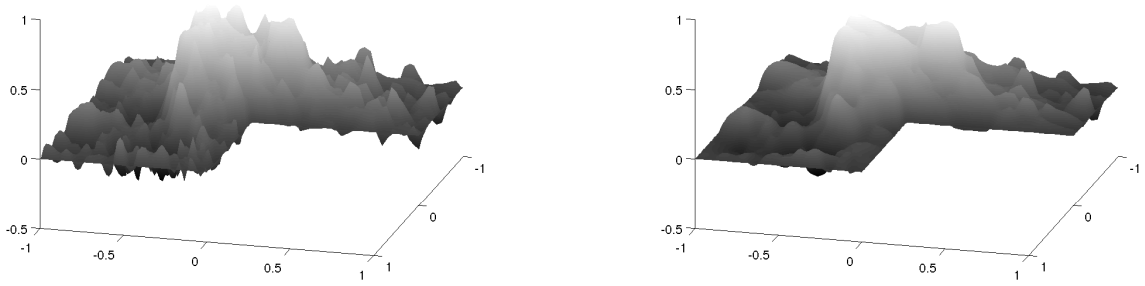
6. Conclusion and Outlook

In this paper, we have studied a new type of random function, which is based on wavelet expansions. With probability one, the realizations have a prescribed regularity in Sobolev as well as in Besov spaces. Essentially, the smoothness of the random function X is controlled by two parameters α and β, where β is a sparsity parameter that can be used to increase the Besov smoothness in scales that determine the order of approximation



(a) exact solution (b) exact right-hand side

FIGURE 6. Equation with unperturbed right-hand side



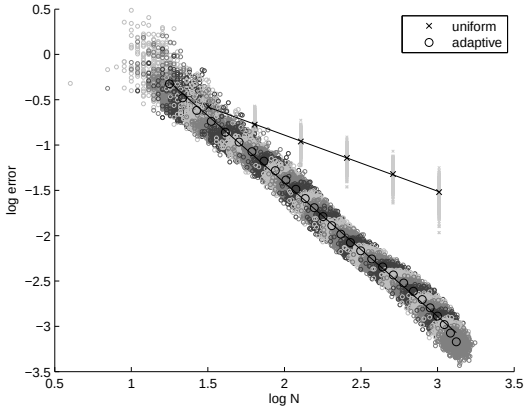
(a) $\alpha = 1.0$, $\beta = 0.1$ (b) $\alpha = 1.0$, $\beta = 0.9$

FIGURE 7. Solution of equation with perturbed right-hand side

for nonlinear wavelet schemes. Beyond the scope of the present paper, our results for X, namely, an explicit control of the Besov regularity and the approximation rates together with an efficient way of simulation, turn this model into an interesting tool to generate test functions in numerical experiments.

Note that X is conditionally Gaussian, given the values of $Y_{j,k}$. Moreover, X can be extended to a time dependent process with values at least in $L_2(D)$, if the normally distributed random variables $Z_{j,k}$ are replaced by independent scalar Brownian motions. In this way we obtain a new kind of a driving process for stochastic partial differential equations of evolutionary type, see [5]. We expect that the results from the present paper will play a significant role in the numerical analysis of these equations.

In our analysis we intend to use the Rothe method, i.e., the evolution equation is first discretized in time and then in space. For stability reasons, one has to use an implicit scheme, so that in each time step an elliptic equation with random right-hand side as discussed in the Sections 3 and 4 has to be solved. To this end, the adaptive wavelet algorithms from Section 4 will be employed.



$\alpha = -48/55$, $\beta = 107/110$

FIGURE 8. Error and (expected) number of non-zero coefficients

The numerical experiments in Section 5 as well as our theoretical analysis presented in this paper indicate that adaptivity really pays off for these problems. Nevertheless, in particular from the univariate examples in Section 5, we still observe a bottleneck. So far, we only discussed random functions with realizations being in smoothness spaces with *positive* smoothness parameters. Then, in the univariate case, the solution immediately possesses Sobolev smoothness larger than two, so that uniform schemes already perform quite well. We expect the picture to change for right-hand sides with *negative* smoothness. Indeed, choosing $X \in H^{-1+\varepsilon}(D)$ yields $U \in H^{1+\varepsilon}(D)$ for $\varepsilon > 0$, and the convergence order of uniform schemes in $H^1(D)$ is only ε. Then, by choosing X in the Besov spaces as in Theorem 3.6, adaptive schemes would still show the same approximation order as before, so that the comparison of uniform and adaptive algorithms would be even more noticeable. Therefore, the generalization of our noise model to spaces of negative smoothness will be studied in a forthcoming paper. We expect that our analysis can be generalized to this setting, since these spaces also possess wavelet characterizations. The following numerical experiments already support our conjecture.

For the moment, let us assume we were allowed to use negative values of α, i.e., Theorem 2.6 carries over to the negative scale. Then, we can choose $\alpha = -48/55$ and $\beta = 107/110$, and we expect the corresponding right-hand side X to be contained in $H^{-0.45}(D)$. In this case, a uniform scheme has approximation order $0.55 - \varepsilon$, which is reflected in Figure 8 an empirical order 0.619. Moreover, let us also assume that Theorem 3.6 carries over, so for adaptive schemes we would still obtain approximation order $1.55 - \varepsilon$. Indeed, Figure 8 shows almost exactly this order, namely 1.476.

Acknowledgement. We are grateful to Natalia Bochkina for valuable discussions, and we thank Tiange Xu for her contributions at an early stage of the project. Also, we thank Winfried Sickel for his comments on regularity theory.

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