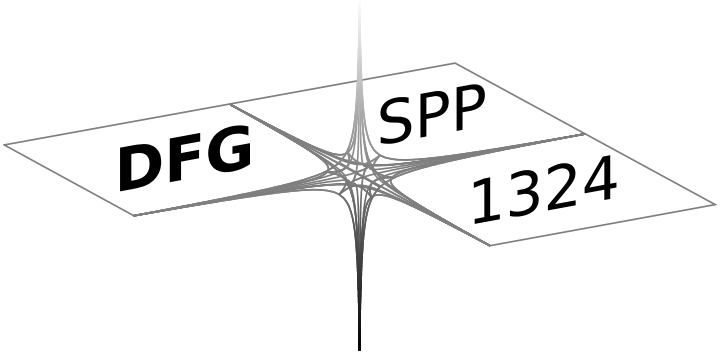
DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Beyond Convergence Rates: Exact Inversion With Tikhonov Regularization With Sparsity Constraints

D.A. Lorenz, S. Schiffler, D. Trede

Preprint 83



Edited by

AG Numerik/Optimierung
Fachbereich 12 - Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Str.
35032 Marburg

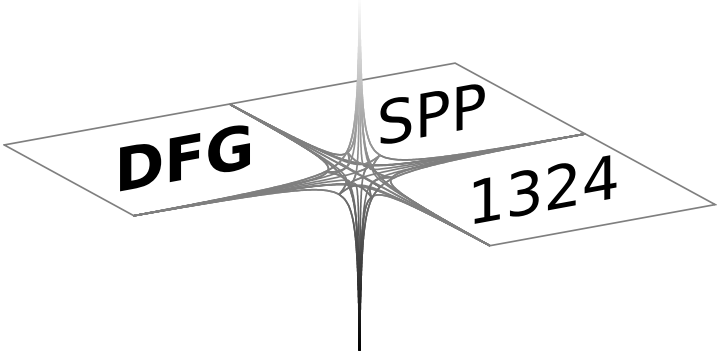
DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen”

Beyond Convergence Rates: Exact Inversion With Tikhonov Regularization With Sparsity Constraints

D.A. Lorenz, S. Schiffler, D. Trede

Preprint 83



The consecutive numbering of the publications is determined by their chronological order.

The aim of this preprint series is to make new research rapidly available for scientific discussion. Therefore, the responsibility for the contents is solely due to the authors. The publications will be distributed by the authors.

BEYOND CONVERGENCE RATES: EXACT INVERSION WITH TIKHONOV REGULARIZATION WITH SPARSITY CONSTRAINTS

DIRK A. LORENZ[†], STEFAN SCHIFFLER[‡], AND DENNIS TREDE[§,¶]

Abstract. The Tikhonov regularization of linear ill-posed problems with an ℓ^1 penalty is considered. Using results from the context of sparse approximation in finite dimensions error estimates in the infinite dimensional case are obtained which are stronger than previous ones. Moreover, conditions which ensure that the regularized solution has the exact support are derived. These conditions are especially applicable in the case of ill-posed problems, where other conditions based on coherence or the restricted isometry property are usually not applicable. The obtained results also show that the regularized solutions do not only converge in the ℓ^1 norm but also in the vector space ℓ^0 (when considered as the strict inductive limit of the spaces $\mathbb{R}^n$ as n tends to infinity).

With an imaging example from digital holography the applicability of the obtained results is illustrated, i.e. that one may check a priori if the experimental setup guarantees exact inversion with Tikhonov regularization with sparsity constraints.

Key words. Tikhonov regularization, sparsity constraints, inverse problems, ill-posed problems, convergence rates, exact recovery, digital holography of particles

AMS subject classifications. 47A52, 65J20

1. Introduction.

In this paper we consider linear inverse problems with a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ between two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$Af = g. \tag{1.1}$$

We are given a noisy observation $g^\varepsilon = g + \eta \in \mathcal{H}_2$ with noise level $\|g - g^\varepsilon\| = \|\eta\| \leq \varepsilon$ and try to reconstruct the solution f of $Af = g$ from the knowledge of g^ε. We are especially interested in the case in which (1.1) is ill-posed in the sense of Nashed, i.e. when the range of A is not closed. In particular this implies that the (generalized) solution of (1.1) is unstable, or in other words, that the generalized inverse $A^\dagger$ is unbounded. In this context, regularization has to be employed to stably solve the problem [12].

We assume that the operator equation $Af = g$ has a solution $f^\diamond$ that can be expressed sparsely in an orthonormal basis $\Psi := \{\psi_i\}_{i\in\mathbb{Z}}$ of $\mathcal{H}_1$, i.e. $f^\diamond$ decomposes into a finite number of basis elements,

$$f^\diamond = \sum_{i\in\mathbb{Z}} u_i^\diamond \psi_i \quad \text{with} \quad u^\diamond \in \ell^2(\mathbb{Z},\mathbb{R}),\ \left|\{i \in \mathbb{Z} \,|\, u_i^\diamond \neq 0\}\right| < \infty.$$

The knowledge that $f^\diamond$ can be expressed sparsely can be utilized for the reconstruction by using an ℓ^1-*penalized Tikhonov* regularization [7], i.e. an approximate solution is

[†]TU Braunschweig, Institute for Analysis and Algebra, Pockelsstr. 14, D-38118 Braunschweig, Germany (d.lorenz@tu-braunschweig.de). Dirk A. Lorenz is supported by the DFG under grant LO 1436/2-1 within the Priority Program SPP 1324 "Extraction of quantitative information in complex systems".

[‡]Zentrum für Technomathematik, Universität Bremen, Fachbereich Mathematik/Informatik, Postfach 33 04 40, D-28334 Bremen, Germany (schiffi@math.uni-bremen.de)

[§]Zentrum für Technomathematik, Universität Bremen, Fachbereich Mathematik/Informatik, Postfach 33 04 40, D-28334 Bremen, Germany (trede@math.uni-bremen.de). Dennis Trede is supported by the BMBF project INVERS under grant 03MAPAH7.

[¶]Author to whom correspondence shall be addressed.

given as a minimizer of the functional

$$\tfrac{1}{2}\|Af - g^\varepsilon\|_{\mathcal{H}_2}^2 + \alpha \sum_{i\in\mathbb{Z}} |\langle f, \psi_i\rangle|, \tag{1.2}$$

with regularization parameter $\alpha > 0$. In contrast to the classical Tikhonov functional with a quadratic penalty [12], the ℓ^1-penalized functional promotes sparsity since small coefficients are penalized more.

For the sake of notational simplification, we introduce the synthesis operator $D : \ell^2 \to \mathcal{H}_1$, which for $u \in \ell^2$ is defined by $Du = \sum u_i \psi_i$. With that and the definition $K := A \circ D : \ell^2 \to \mathcal{H}_2$ we can rewrite the inverse problem (1.1) as $Ku = g$ and the ℓ^1-penalized Tikhonov regularization (1.2) as

$$T_\alpha(u) := \tfrac{1}{2}\|Ku - g^\varepsilon\|_{\mathcal{H}_2}^2 + \alpha\|u\|_{\ell^1}. \tag{1.3}$$

In the following we frequently use the standard basis of ℓ^2, which is denoted by $\{e_j\}_{j\in\mathbb{Z}}$.

The Tikhonov functional (1.3) has also been used in the context of sparse recovery under the name Basis Pursuit Denoising [5].

In this paper we analyze the regularization properties of (1.3). Daubechies et al. [7] showed that the minimization of (1.3) is indeed a regularization while error estimates and convergence rates have been derived by Lorenz [20] and Grasmair et al. [15] under additional source conditions. In this paper we derive conditions that ensure that the minimizers $u^{\alpha,\varepsilon} \in \arg\min T_\alpha(u)$ have the same support as $u^\diamond$. In the context of sparse recovery, this phenomenon is called *exact recovery*. Our contribution is that we transfer results on exact recovery from [11, 13, 14, 17, 25] to the case of ill-posed linear operator equations. The field of sparse recovery and ill-posed linear operator equations differ in several points. First, sparse recovery is formulated in finite dimensional space while ill-posed linear operator equations are formulated in infinite dimensional Banach and Hilbert spaces. As a consequence, different topologies can be considered (besides the weak and the strong topology of ℓ^2, also the topologies of ℓ^1 and ℓ^∞ are meaningful in our case). Moreover, sparse recovery usually assumes an overcomplete dictionary for modeling the signal. The quality of the dictionary is expressed in terms of incoherence. The emphasis is on highly overcomplete dictionaries with small coherence. In ill-posed linear operator equations redundancy is often not needed but the problem comes from ill-posedness in the sense of arbitrarily small singular values. Unless the dictionary is close to the singular vectors of the operator, this makes coherence concepts in general not applicable and one has to resort to other conditions [9, 11].

The paper is organized as follows. In §2 we summarize some properties of ℓ^1-penalized Tikhonov minimizers and recall a stability result from [15]. In §3 we review previous results on exact recovery in the context of sparse recovery and we illustrate our contribution. Section 4 contains the main theoretical results of the paper, namely, the transfer of the results from [11,13,14,17,25] reviewed in §3 to the case of ill-posed linear operator equations. In §5, we demonstrate the practicability of the deduced recovery condition with an example from imaging, namely, an example from digital holography. In §6 we give a conclusion on exact recovery conditions for Tikhonov regularization with sparsity constraints.

2. The ℓ^1-penalized Tikhonov functional. Before we start with error estimates, we recall some basic properties of the ℓ^1-penalized Tikhonov functional T_α. First we give a characterization of the minimizer.

PROPOSITION 2.1 (Optimality condition). *Define the* set-valued sign function $\operatorname{Sign} : \ell^2 \to \big\{\{-1\},[-1,+1],\{+1\}\big\}^{\mathbb{Z}}$, *for* $u \in \ell^2$, *by*

$$\big(\operatorname{Sign}(u)\big)_k := \begin{cases} \{-1\}, & u_k < 0, \\ [-1,+1], & u_k = 0, \\ \{+1\}, & u_k > 0. \end{cases} \tag{2.1}$$

Let $u^{\alpha,\varepsilon} \in \ell^2$. *Then the following statements are equivalent:*

(i) $$u^{\alpha,\varepsilon} \in \operatorname*{arg\,min}_{u\in\ell^2} \tfrac{1}{2}\|Ku - g^\varepsilon\|_{\mathcal{H}_2}^2 + \alpha\|u\|_{\ell^1}. \tag{2.2}$$

(ii) $$-K^*(Ku^{\alpha,\varepsilon} - g^\varepsilon) \in \alpha \operatorname{Sign}(u^{\alpha,\varepsilon}). \tag{2.3}$$

Notice that this characterization is obvious, since the set-valued sign function is actually the subgradient of the ℓ^1 norm. Another well known characterization of a minimizer $u^{\alpha,\varepsilon}$ is, that it is a fixed point of $u^{\alpha,\varepsilon} = \mathbb{S}_\alpha(u^{\alpha,\varepsilon} + K^*(g^\varepsilon - Ku^{\alpha,\varepsilon}))$, where $\mathbb{S}_\alpha$ denotes the soft-thresholding operator, cf. e.g. [7]. From this characterization or from (2.3) we can deduce the following: Since the range of K^* is contained in ℓ^2, any minimizer $u^{\alpha,\varepsilon}$ of the ℓ^1-penalized Tikhonov functional T_α is finitely supported for every $\alpha > 0$. (Note that this observation relies on the fact that K is bounded on ℓ^2. If we model $K : \ell^1 \to \mathcal{H}_2$ boundedly, as appropriate for normalized dictionaries, we cannot conclude that $u^{\alpha,\varepsilon}$ is finitely supported.)

Uniqueness of the minimizer of (1.3) could be ensured by ensuring strict convexity. This holds, e.g., if K is injective. A weaker property of the operator K, which also guarantees uniqueness (although the functional is not strictly convex) is the FBI [1] property defined below.

DEFINITION 2.2. *Let* $K : \ell^2 \to \mathcal{H}$ *be an operator mapping into a Hilbert space* $\mathcal{H}$. *Then* K *has the* finite basis injectivity (FBI) *property, if for all finite subsets* $J \subset \mathbb{Z}$ *the operator restricted to* J *is injective, i.e. for all* $u, v \in \ell^2$ *with* $Ku = Kv$ *and* $u_k = v_k = 0$, *for all* $k \notin J$, *it follows that* $u = v$.

In inverse problems with sparsity constraints the FBI property is used for a couple of theoretical results for the ℓ^1-penalized Tikhonov functional, cf. e.g. [6,15,16,18–20]. A demonstrative example for an operator which possesses the FBI property but is not fully injective is the synthesis operator which maps wavelet and Fourier coefficients to a signal which is superposition of wavelets and complex exponentials.

The FBI property is related to the so-called *restricted isometry property (RIP)* [4] of a matrix, which is a quite common assumption in the theory of compressive sampling [2, 3]. The RIP is defined as follows. Let A be a $m \times n$ matrix and let $s < n$ be an integer. The *restricted isometry constant* of order s is defined as the smallest number $0 < c_s < 1$, such that the following condition holds for all $v \in \mathbb{R}^n$ with at most s non-zero entries:

$$(1 - c_s)\|v\|_{\ell^2}^2 \le \|Av\|_{\ell^2}^2 \le (1 + c_s)\|v\|_{\ell^2}^2.$$

The matrix A has the RIP if the constants c_s stay reasonably small for an adequate range of s. Essentially, this property denotes that the matrix is approximately an isometry when restricted to small subspaces. The FBI property, however, is defined for operators acting on the sequence space and only says, that the restriction to finite dimensional subspaces is still injective and makes no assumption of the involved constants.

With ℓ^0 we denote the vector space of all real-valued sequence with only finitely many non-zero entries. In contrast to the ℓ^p spaces with $p > 0$ there is no obvious (quasi-)norm available which turns ℓ^0 into a (quasi-)Banach space. We will come back to the issue of defining a suitable topology on ℓ^0 later. In general, the minimum-$\|\cdot\|_{\ell^1}$ solution $u^\dagger$ of $Ku = g$ neither needs to be in ℓ^0, nor needs to be unique. If we assume that there is a finitely supported solution $u^\diamond \in \ell^0$ of $Ku = g$, then the set of all solutions of $Ku = g$ is given by $u^\diamond + \ker K$. If K possesses the FBI property, then the solution $u^\diamond$ is the unique solution in ℓ^0, hence $\ker K \subset \ell^2 \setminus \ell^0$. However, in general $u^\diamond \in \ell^0$ is not a minimum-$\|\cdot\|_{\ell^1}$ solution. In the following we assume that K possesses the FBI property and denote the unique solution of $Ku = g$ in ℓ^0 with $u^\diamond$ and a minimum-$\|\cdot\|_{\ell^1}$ solution of $Ku = g$ with $u^\dagger$.

Stability and convergence rates results for ℓ^1-penalized Tikhonov functionals have been deduced in [7,15,16,20]. The following error estimate from [15] ensures the linear convergence to the minimum-$\|\cdot\|_{\ell^1}$ solution $u^\dagger$, if a certain source condition is satisfied. We state it here in full detail and give explicit constants.

THEOREM 2.3 (Error estimate [15, theorem 15]). *Let K possess the FBI property, $u^\dagger \in \ell^0$ with $\operatorname{supp} u^\dagger = I$ be a minimum-$\|\cdot\|_{\ell^1}$ solution of $Ku = g$, and $\|g - g^\varepsilon\|_{\mathcal{H}_2} \le \varepsilon$. Let the following source condition be fulfilled:*

$$\text{there exists } w \in \mathcal{H}_2 \text{ such that } K^* w = \xi \in \operatorname{Sign}(u^\dagger). \tag{2.4}$$

Moreover, let

$$\theta = \sup\left\{ |\xi_k| \,\middle|\, |\xi_k| < 1 \right\} < 1$$

and $c > 0$ such that for all $u \in \ell^2$ with $\operatorname{supp}(u) \subset I$ it holds

$$\|Ku\| \ge c\|u\|.$$

Then for the minimizers $u^{\alpha,\varepsilon}$ of

$$T_\alpha(u) = \tfrac{1}{2}\|Ku - g^\varepsilon\|_{\mathcal{H}_2}^2 + \alpha\|u\|_{\ell^1}$$

it holds

$$\|u^{\alpha,\varepsilon} - u^\dagger\|_{\ell^1} \le \frac{\|K\| + 1}{1 - \theta}\frac{\varepsilon^2}{\alpha} + \Big(\frac{1}{c} + \|w\|\frac{\|K\| + 1}{1 - \theta}\Big)(\alpha + \varepsilon). \tag{2.5}$$

Especially, with $\alpha \asymp \varepsilon$ it holds

$$\|u^{\alpha,\varepsilon} - u^\dagger\|_{\ell^1} = \mathcal{O}(\varepsilon). \tag{2.6}$$

REMARK 1.

a) Since $\xi \in \ell^2$ by definition of K, it is clear that $\theta < 1$.
b) Since K possesses the FBI property, the existence of $c > 0$ is ensured.
c) To achieve the linear convergence rate (2.6), the source condition (2.4) is even necessary, cf. [16].

The above theorem is remarkable since it gives an error estimate for regularization with a sparsity constraints with comparably weak conditions of the operator, especially nothing is assumed about the incoherence of K in either way. However, the constants in the error estimate (2.5) are both depending on the unknown quantities I, ξ and θ and are possibly huge (especially c can be small and θ can be close to one).

3. Known results from sparse recovery. The results presented in the next section build upon results from [13, 14] and [25]. In [13], Fuchs gives a condition, which ensures that the support of the Tikhonov minimizer with noiseless data g, and the support of $u^\diamond \in \ell^0$ coincide. In [14], Fuchs transfers his results from [13] to noisy signals. Assuming that the so-called coherence parameter $\mu := \sup_{i \neq j} |\langle Ke_i, Ke_j \rangle|$ is small, he proves a condition which ensures $\operatorname{supp}(u^{\alpha,\varepsilon}) = \operatorname{supp}(u^\diamond)$. Unless the dictionary is close to the singular vectors of the operator, for ill-posed inverse problems the coherence parameter μ is typically huge (i.e. $\mu \approx 1$) and hence Fuchs' results cannot be used.

Independently, in [25] Tropp deduces another condition which ensures exact recovery without using the assumption of a small coherence parameter. To formulate the statement, we need the following notations. For a subset $J \subset \mathbb{Z}$, we denote with $P_J : \ell^2 \to \ell^2$ the projection onto J,

$$P_J u := \textstyle\sum_{j \in J} u_j e_j,$$

i.e. the coefficients $j \notin J$ are set to 0 and hence $\operatorname{supp}(P_J u) \subset J$. With that definition $KP_J : \ell^2 \to \mathcal{H}_2$ denotes the operator K restricted to J. Moreover, for a linear operator B we denote the pseudoinverse operator by $B^\dagger$. With these definitions we are able to formulate Tropp's condition for exact recovery.

Let K be injective on I, and let g_I^ε be the orthogonal projection of g^ε to the range of KP_I. Then there is a unique $u_I^\varepsilon \in \ell^2$ with $\operatorname{supp}(u_I^\varepsilon) \subset I$ so that $g_I^\varepsilon = K u_I^\varepsilon$. On the *exact recovery condition*

$$\sup_{i \in I^\complement} \|(KP_I)^\dagger K e_i\|_{\ell^1} < 1,$$

Theorem 8 from [25] gives the following parameter choice rule which ensures that $\operatorname{supp}(u^{\alpha,\varepsilon}) = \operatorname{supp}(u^\diamond)$:

$$\frac{\sup_{i \in I^\complement} |\langle g^\varepsilon - g_I^\varepsilon, Ke_i \rangle|}{1 - \sup_{i \in I^\complement} \|(KP_I)^\dagger K e_i\|_{\ell^1}} < \alpha < \frac{\min_{i \in I} |u_I^\varepsilon(i)|}{\|(P_I K^* K P_I)^{-1}\|_{\ell^1, \ell^1}}. \tag{3.1}$$

The applicability of this result is limited due to several terms: The expressions $\|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}$ and $\|(KP_I)^\dagger K e_i\|_{\ell^1}$, need the knowledge of I which is unknown. Moreover, the quantities g_I^ε (the projection of g^ε onto the range of KP_I) and $u_I^\varepsilon = K g_I^\varepsilon$ are unknown and not computable without the knowledge of I.

In the following section we transfer these results to the case of ill-posed linear operator equations and deduce a-priori parameter rules that are easier to use than Tropp's parameter choice rule (3.1) from [25]. The idea is to get rid of the expressions g_I^ε and u_I^ε, which cannot be estimated a priori. Furthermore, we will apply the techniques from [11,17] to deal with the terms $\|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}$ and $\|(KP_I)^\dagger K e_i\|_{\ell^1}$.

4. Beyond convergence rates: exact recovery for ill-posed operator equations. In this paragraph we give an a priori parameter rule which ensures that the unknown support of the sparse solution $u^\diamond \in \ell^0$ is recovered exactly, i.e. $\operatorname{supp}(u^{\alpha,\varepsilon}) = \operatorname{supp}(u^\diamond)$. We assume that K possesses the FBI property, and hence $u^\diamond$ is the unique solution of $Ku = g$ in ℓ^0. With I we denote the support of $u^\diamond$, i.e.

$$I := \operatorname{supp}(u^\diamond) := \{i \in \mathbb{Z} \,|\, u_i^\diamond \neq 0\}.$$

THEOREM 4.1 (Lower bound on α). *Let $u^\diamond \in \ell^0$, $\operatorname{supp}(u^\diamond) = I$, and $g^\varepsilon = Ku^\diamond + \eta$ the noisy data with noise level $\|\eta\|_{\mathcal{H}_2} \leq \varepsilon$. Assume that the operator norm of K is*

bounded by 1 and that K possesses the FBI property. If the following condition holds,

$$\sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1} < 1, \tag{4.1}$$

then the parameter rule

$$\alpha > \frac{1+\sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1}}{1-\sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1}} \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle| \tag{4.2}$$

ensures that the support of $u^{\alpha,\varepsilon} := \arg\min T_\alpha(u)$ is contained in I.

Proof. In [25, theorem 8] it is shown that condition (4.1) together with the parameter rule

$$\alpha > \frac{\sup_{i\in I^{\complement}} |\langle g^\varepsilon - g_I^\varepsilon, Ke_i\rangle|}{1-\sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1}}$$

ensures that the support of $u^{\alpha,\varepsilon} = \arg\min T_\alpha(u)$ is contained in I. Here, g_I^ε be the orthogonal projection of g^ε to the range of KP_I, i.e. $g_I^\varepsilon = KP_I(KP_I)^\dagger g^\varepsilon$. Since $\mathrm{Id} - KP_I(KP_I)^\dagger$ equals the orthogonal projection on $\mathrm{rg}(KP_I)^\perp$ and $g = Ku^\diamond \in \mathrm{rg}(KP_I)$, we get

$$g^\varepsilon - g_I^\varepsilon = \big(\mathrm{Id} - KP_I(KP_I)^\dagger\big)(g^\varepsilon - g) = \big(\mathrm{Id} - KP_I(KP_I)^\dagger\big)(\eta).$$

Hence, with Hölder's inequality, for $i \in I^{\complement}$ it holds that

$$\begin{aligned}
|\langle g^\varepsilon - g_I^\varepsilon, Ke_i\rangle| &= |\langle (\mathrm{Id} - KP_I(KP_I)^\dagger)\,\eta, Ke_i\rangle| \\
&= |\langle \eta, (\mathrm{Id} - KP_I(KP_I)^\dagger) Ke_i\rangle| \\
&= |\langle K^*\eta, (\mathrm{Id} - (KP_I)^\dagger K) e_i\rangle_{\ell^2}| \\
&\le \|K^*\eta\|_{\ell^\infty}\, \|(\mathrm{Id} - (KP_I)^\dagger K) e_i\|_{\ell^1}.
\end{aligned}$$

Since $i \in I^{\complement}$ and $\mathrm{supp}((KP_I)^\dagger K e_i) \subset I$ we get

$$\sup_{i\in I^{\complement}} \|e_i - (KP_I)^\dagger K e_i\|_{\ell^1} = 1 + \sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1}.$$

Hence, we end in the estimate

$$\sup_{i\in I^{\complement}} |\langle g^\varepsilon - g_I^\varepsilon, Ke_i\rangle| \le \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle| \big(1 + \sup_{i\in I^{\complement}} \|(KP_I)^\dagger K e_i\|_{\ell^1}\big). \tag{4.3}$$

Thus, the condition (4.1) together with the parameter choice rule (4.2) ensures that the support is contained in I. A direct proof without using [25] can be found in [24]. □

Remark 2. The assumption $\|K\| \le 1$ has been introduced for the sake of notational simplification. However, afterwards we will need it for some estimates. The assumption that K is an operator with norm bounded by 1 is a common condition for Tikhonov functionals with sparsity constraints, cf. e.g. [7]. Actually, this is not really a limitation, because every inverse problem $Ku = g$ with bounded operator K can be rescaled so that $\|K\| \le 1$ holds.

Remark 3. Instead of using the estimate (4.3), one can alternatively use another upper bound for $\sup_{i\in I^{\complement}} |\langle g^\varepsilon - g_I^\varepsilon, Ke_i\rangle|$. For that notice that $\mathrm{Id} - KP_I(KP_I)^\dagger$ is the

orthogonal projection on $\operatorname{rg}(KP_I)^\perp$. Then, since the norm of K and the norm of orthogonal projections are bounded by 1, we can estimate for $i \in I^\complement$ with the Cauchy-Schwarz inequality as follows

$$|\langle \eta, (\mathrm{Id} - KP_I(KP_I)^\dagger) Ke_i \rangle| \leq \|\eta\|_{\mathcal{H}_2} \|(\mathrm{Id} - KP_I(KP_I)^\dagger Ke_i\|_{\mathcal{H}_2} \leq \varepsilon. \tag{4.4}$$

In general, one cannot say which estimate gives a sharper bound, inequality (4.3) or inequality (4.4). However, in practice the noise η often is uniformly distributed and hence $\sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle| \ll \varepsilon$ holds. In this case the estimate with Hölder's inequality (4.3) gives a sharper estimate. We use (4.3) for the example from digital holography in §5.

Theorem 4.1 gives a lower bound on the regularization parameter α to ensure $\operatorname{supp}(u^{\alpha,\varepsilon}) \subset \operatorname{supp}(u^\diamond)$. To guarantee $\operatorname{supp}(u^{\alpha,\varepsilon}) = \operatorname{supp}(u^\diamond)$ we need an additional upper bound for α. The following theorem leads to that purpose.

Theorem 4.2 (Error estimate). *Let the assumptions of Theorem 4.1 hold. Let K and I fulfill (4.1) and choose α according to (4.2), i.e. it holds that* $\operatorname{supp}(u^{\alpha,\varepsilon}) \subset I$. *Then the following error estimate is valid:*

$$\|u^\diamond - u^{\alpha,\varepsilon}\|_{\ell^\infty} \leq (\alpha + \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|) \|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}. \tag{4.5}$$

Proof. From the assumptions of Theorem 4.1 we have $\operatorname{supp}(u^{\alpha,\varepsilon}) \subset \operatorname{supp}(u^\diamond)$. From the optimality condition (2.3) we know that for $u^{\alpha,\varepsilon}$ there is a $w \in \ell^\infty$ with $\|w\|_{\ell^\infty} \leq 1$ such that

$$-K^*(Ku^{\alpha,\varepsilon} - g^\varepsilon) = \alpha\, w.$$

Hence, it holds that

$$\begin{aligned} -P_I K^* K P_I (u^{\alpha,\varepsilon} - u^\diamond) &= -P_I K^*(Ku^{\alpha,\varepsilon} - g) = -P_I K^*(Ku^{\alpha,\varepsilon} - g^\varepsilon) - P_I K^* \eta \\ &= \alpha P_I w - P_I K^* \eta. \end{aligned}$$

Since $\|w\|_{\ell^\infty} \leq 1$, with Hölder's inequality we can estimate for all $j \in I$

$$\begin{aligned} |(u^{\alpha,\varepsilon} - u^\diamond)_j| = |\langle u^{\alpha,\varepsilon} - u^\diamond, e_j\rangle| &= \left|\left\langle (P_I K^* K P_I)^{-1}\big(\alpha P_I w - P_I K^* \eta\big), e_j \right\rangle\right| \\ &\leq \alpha |\langle P_I w, (P_I K^* K P_I)^{-1} e_j\rangle| + |\langle P_I K^* \eta, (P_I K^* K P_I)^{-1} e_j\rangle| \\ &\leq \big(\alpha \|w\|_{\ell^\infty} + \|P_I K^* \eta\|_{\ell^\infty}\big) \|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1} \\ &\leq \big(\alpha + \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|\big) \|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}. \end{aligned}$$

□

Remark 4. Due to the error estimate (4.5) we achieve a linear convergence rate measured in the ℓ^∞ norm. In finite dimensions the ℓ^p norms are equivalent, hence we also get an estimate for the ℓ^1 error:

$$\|u^\diamond - u^{\alpha,\varepsilon}\|_{\ell^1} \leq (\alpha + \varepsilon)\ |I|\ \|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}.$$

Compared to the estimate (2.5) from Theorem 2.3, the quantities θ and $\|w\|$ are not present anymore. The role of $1/c$ is now played by $\|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}$. However, if upper bounds on I or on its size (together with structural information on K is available), our estimate can give a-priori checkable error estimates.

The following theorem gives a sufficient condition for the existence of a regularization parameter α which provides exact recovery. Due to Theorem 4.2, equation (4.5), the regularization parameter should be chosen as small as possible.

THEOREM 4.3 (Exact recovery condition in the presence of noise). *Let $u^\diamond \in \ell^0$ with $\operatorname{supp}(u^\diamond) = I$ and $g^\varepsilon = Ku^\diamond + \eta$ the noisy data with noise level $\|\eta\|_{\mathcal{H}_2} \leq \varepsilon$ and noise-to-signal ratio*

$$r_{\varepsilon/u} := \frac{\sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|}{\min_{i\in I} |u_i^\diamond|}.$$

Assume that the operator norm of K is bounded by 1 and that K possesses the FBI property. Then the exact recovery condition in the presence of noise (εERC)

$$\sup_{i\in I^\complement} \|(KP_I)^\dagger Ke_i\|_{\ell^1} < 1 - 2r_{\varepsilon/u}\|(P_IK^*KP_I)^{-1}\|_{\ell^1,\ell^1} \tag{4.6}$$

ensures that there is a suitable regularization parameter α,

$$\begin{aligned} \frac{1+\sup_{i\in I^\complement}\|(KP_I)^\dagger Ke_i\|_{\ell^1}}{1-\sup_{i\in I^\complement}\|(KP_I)^\dagger Ke_i\|_{\ell^1}} \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle| &< \alpha \\ \alpha &< \frac{\min_{i\in I}|u_i^\diamond|}{\|(P_IK^*KP_I)^{-1}\|_{\ell^1,\ell^1}} - \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|, \end{aligned} \tag{4.7}$$

which provides exact recovery of I, i.e. the support of the minimizer $u^{\alpha,\varepsilon}$ coincides with $\operatorname{supp}(u^\diamond) = I$.

Proof. The lower bound of the parameter rule (4.7) ensures that $\operatorname{supp}(u^{\alpha,\varepsilon}) \subset I$. With the error estimate (4.5) we see that the upper bound from the parameter rule (4.7) guarantees for $j \in I$ that

$$|u_j^\diamond| - |u_j^{\alpha,\varepsilon}| \leq |u_j^\diamond - u_j^{\alpha,\varepsilon}| < \min_{i\in I}|u_i^\diamond|,$$

hence $|u_j^{\alpha,\varepsilon}| > |u_j^\diamond| - \min_{i\in I}|u_i^\diamond| \geq 0$ for all $j \in I$. The εERC (4.6) ensures that the interval of convenient regularization parameters α resulting from (4.7) is not empty. ∎

The results of Theorem 4.2 and 4.3 can be rephrased as follows: If the regularization parameter $\alpha(\varepsilon)$ is chosen according to (4.7) and fulfills $\alpha \asymp \varepsilon$, then $\|u^{\alpha,\varepsilon} - u^\diamond\|_{\ell^1} = \mathcal{O}(\varepsilon)$ and the support of $u^{\alpha,\varepsilon}$ coincides with that of $u^\diamond$. This can be interpreted as convergence in the space ℓ^0 with respect to the following topology.

DEFINITION 4.4. *We equip the spaces $\mathbb{R}^n$ with the Euclidean topology and consider them ordered by inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ in the natural way. Then an absolutely convex and absorbent subset U of ℓ^0 is called a neighborhood of 0 if set $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for any n. The topology τ on ℓ^0 which is generated by the local base of these neighborhoods is called the* topology of sparse convergence.

The definition above says that the topology of sparse convergence is generated as strict inductive limit of the spaces $\mathbb{R}^n$. The space ℓ^0 is turned into a complete locally convex vector space with this topology. A sequence (u^n) in ℓ^0 converges to u in this topology if there is a finite set $I \subset \mathbb{N}$ such that $\operatorname{supp} u^n \subset I$, for all $n \in \mathbb{N}$, and the sequence u^n converges componentwise. As a strict inductive limit of Fréchet spaces, (ℓ^0, τ) is also called an LF-space and is known to be not normable, see [10, 21]. The topology of sparse convergence resembles the topology on the space of test functions $\mathcal{D}(\Omega)$ in distribution theory. (This correspondence can be pushed a little bit further by observing that the dual space of (ℓ^0, τ) is the space of all real valued sequences and

plays the rule of the space of distributions. We will not pursue this similarity further here.)

COROLLARY 4.5 (Convergence in ℓ^0). *In the situation of Theorem 4.3 assume that α is chosen to fulfill the parameter choice rule* (4.7). *Then $u^{\alpha,\varepsilon} \to u^\diamond$ in ℓ^0 in the topology of sparse convergence.*

In fact, the parameter choice rule (4.7) is not an a priori parameter rule $\alpha = \alpha(\varepsilon)$, since it depends on the noise η and on unknown quantities such as I and $\min_i |u_i^\diamond|$. However, the term $\sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|$ is related to the noise level and it can be estimated by ε, cf. Remark 3. The term $\min_i |u_i^\diamond|$, i.e. the smallest non-zero entry in the unknown solution may be estimated from below in several application. Due to the expressions $\|(P_I K^* K P_I)^{-1}\|_{\ell^1,\ell^1}$ and $\sup_{i\in I^\complement} \|(KP_I)^\dagger Ke_i\|_{\ell^1}$, the εERC (4.6) is hard to evaluate, especially since the support I is unknown. Therefore, we follow [11] and give a weaker sufficient recovery condition, that depends on inner products of images of K restricted to I and $I^\complement$. For the sake of an easier presentation we define according to [11, 17]

$$\mathrm{COR}_I := \sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle Ke_i, Ke_j\rangle| \qquad \text{and} \qquad \mathrm{COR}_{I^\complement} := \sup_{i\in I^\complement} \sum_{j\in I} |\langle Ke_i, Ke_j\rangle|.$$

THEOREM 4.6 (Neumann exact recovery condition in the presence of noise). *Let $u^\diamond \in \ell^0$ with $\operatorname{supp}(u^\diamond) = I$ and $g^\varepsilon = Ku^\diamond + \eta$ the noisy data with noise level $\|\eta\|_{\mathcal{H}_2} \leq \varepsilon$ and noise-to-signal ratio $r_{\varepsilon/u}$. Assume that the operator norm of K is bounded by 1 and that K possesses the FBI property. Then the* Neumann exact recovery condition in the presence of noise (Neumann εERC)

$$\mathrm{COR}_I + \mathrm{COR}_{I^\complement} < \min_{i\in I} \|Ke_i\|_{\mathcal{H}_2}^2 \; - \; 2r_{\varepsilon/u} \tag{4.8}$$

ensures that there is a suitable regularization parameter α,

$$\frac{\min_{i\in I} \|Ke_i\|_{\mathcal{H}_2}^2 - \mathrm{COR}_I + \mathrm{COR}_{I^\complement}}{\min_{i\in I} \|Ke_i\|_{\mathcal{H}_2}^2 - \mathrm{COR}_I - \mathrm{COR}_{I^\complement}} \; \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle| < \alpha \tag{4.9}$$

$$\alpha < \big(\min_{i\in I} \|Ke_i\|_{\mathcal{H}_2}^2 - \mathrm{COR}_I\big) \min_{i\in I} |u_i^\diamond| - \sup_{i\in\mathbb{Z}} |\langle \eta, Ke_i\rangle|,$$

which provides exact recovery of I, i.e. the support of $u^{\alpha,\varepsilon}$ coincides with $\operatorname{supp}(u^\diamond) = I$.

Proof. For the deduction of conditions (4.8) and (4.9) from conditions (4.6) and (4.7), respectively, one can split the operator $P_I K^* K P_I$ into diagonal and off-diagonal and use Neumann series expansion for its inverse, following the techniques from [11, 17]. □

REMARK 5. By the assumptions of Theorem 4.6, the operator norm of K is bounded by 1, i.e. $\|Ke_i\|_{\mathcal{H}_2} \leq 1$ for all $i \in \mathbb{Z}$. Hence, to ensure the Neumann εERC (4.8), one has necessarily for the noise-to-signal ratio $r_{\varepsilon/u} < 1/2$. For a lot of examples one can normalize K, so that $\|Ke_i\|_{\mathcal{H}_2} = 1$ holds for all $i \in \mathbb{Z}$. We do this for the example from digital holography in §5. In this case the Neumann εERC (4.8) reads as

$$\mathrm{COR}_I + \mathrm{COR}_{I^\complement} < 1 \; - \; 2r_{\varepsilon/u}.$$

This condition coincides with the result presented in [9] for the orthogonal matching pursuit.

REMARK 6. We remark that the correlations COR_I and $\mathrm{COR}_{I^\complement}$ can be estimated from above, with $N := \|u^\diamond\|_{\ell^0}$, by

$$\mathrm{COR}_I \leq (N-1)\,\mu \qquad \text{and} \qquad \mathrm{COR}_{I^\complement} \leq N\,\mu.$$

Consequently, the exact recovery condition in terms of the coherence parameter μ can easily be deduced from conditions (4.8) and (4.9). This will result in Fuchs' exact recovery condition from [14].

5. Application of exact recovery conditions to digital holography. To apply the Neumann εERC (4.8), one has to know the support I. In this case, there is no need to apply complex reconstruction methods. One may just solve the restricted least squares problem. For deconvolution problems, however, with a certain prior knowledge, it is possible to evaluate the Neumann εERC (4.8) a priori, especially when the support I is not known exactly.

In the following we use the Neumann εERC (4.8) exemplarily for characteristic functions convolved with a Fresnel function, as it is e.g. used in digital holography of particles. For the sake of an easier presentation, we normalize the operator K such that $\|Ke_i\|_{\mathcal{H}_2} = 1$ for all $i \in \mathbb{Z}$.

Holography is a technique invented by Dennis Gabor in 1948 for recording and reconstructing amplitude and phase of a wave field. The word holography is composed of the Greek words "holos" meaning whole or entire and "graphein" meaning to write.

It is a two-step technique. First, an interference pattern between a wave field scattered from an object and a reference wave is recorded—the so-called hologram. The (two-dimensional) hologram contains the information about the entire three-dimensional wave field, coded in terms of interference stripes. The second step is the reconstruction of the object wave by illuminating with the reference wave again, which creates a three-dimensional image.

In *digital* holography, the hologram is recorded digitally on a charge-coupled device (CCD), and reconstructed numerically. There are numerous applications of digital holography, e.g. in material engineering and biological imaging. In this paper we deal with another application: three-dimensional particle tracking in fluid mechanics. Here, the size and the distribution of particles are extracted from a hologram.

In digital holography, the data correspond to the diffraction patterns of the objects. Under Fresnel's approximation, diffraction can be modeled by a convolution with a "chirp" kernel. In the context of holograms of particles, the objects can be considered opaque (i.e. binary), and the hologram recorded on the camera corresponds to the convolution of disks with Fresnel's chirp kernels. The measurement of particle size and location therefore amounts to an inverse convolution problem [8, 22].

We consider the case of spherical particles, which is of significant interest in applications such as fluid mechanics. We model the particles $j \in \{1, \ldots, N\}$ as opaque disks $B_r(\cdot - x_j, \cdot - y_j, \cdot - z_j)$ with center $(x_j, y_j, z_j) \in \mathbb{R}^3$, radius r and disk orientation orthogonal to the optical axis (Oz). Hence the source $f^\diamond$ is given as a sum of characteristic functions

$$f^\diamond = \sum_{j=1}^{N} u_j^\diamond\, \chi_{B_r}(\cdot - x_j, \cdot - y_j, \cdot - z_j) =: \sum_{j=1}^{N} u_j^\diamond\, \chi_j.$$

The real values $u_j^\diamond$ are amplitude factors of the diffraction pattern that in practice depend on experimental parameters.

The forward operator $K : \ell^2 \to L^2(\mathbb{R}^2)$, which maps the coefficients u_j to the corresponding digital hologram, is well modeled by a bidimensional convolution $*$ with

respect to (x, y). In the following ι represents the imaginary unit. Let h_{z_j} constitute the Fresnel function defined by

$$h_{z_j}(x,y) = \frac{1}{\iota\lambda z_j} \exp\Big(\iota\frac{\pi}{\lambda z_j}\|R\|^2\Big), \qquad \text{with } R := (x,y).$$

With that, the hologram of a particle at position (x_j, y_j, z_j) and hence the corresponding operator response Ke_j has the following form [22]:

$$\big(Ke_j\big)(x,y) := \frac{2}{\pi r^2}\chi_{B_r}(x-x_j, y-y_j) \;*\; \mathrm{Re}\big(h_{z_j}(x-x_j, y-y_j)\big). \tag{5.1}$$

The factor $2/(\pi r^2)$ assures Ke_j to be unit-normed, cf. [9].

In practice, the function χ_{B_r} has a bounded and small support with respect to $\sqrt{\lambda z}$—the so-called far-field condition. Let J_1 denote the first kind Bessel function of order one. In [9] it is shown, that if the far-field condition is fulfilled, then we can represent Ke_j with $R_j := (x - x_j, y - y_j)$ by

$$\big(Ke_j\big)(x,y) \approx \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \frac{1}{\|R_j\|} J_1\Big(\frac{2\pi r}{\lambda z_j}\|R_j\|\Big) \sin\Big(\frac{\pi}{\lambda z_j}\|R_j\|^2\Big). \tag{5.2}$$

In figure 5.1 the hologram of a particle at the origin, i.e. Ke_j for $(x_j, y_j) = (0, 0)$, with a radius of 50µm is displayed (illuminated with a red laser beam (wavelength 630nm) in a distance of z_j =250mm to the camera).

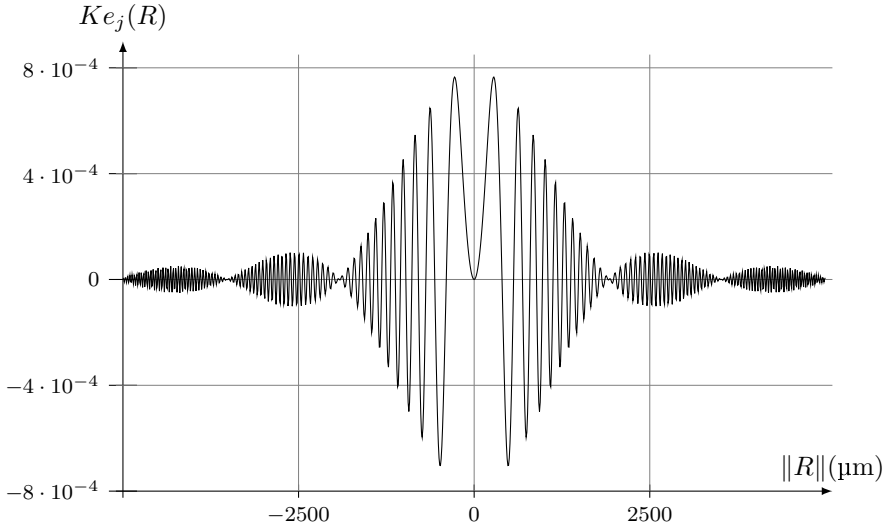


FIG. 5.1. *Digital hologram of a particle of radius 50µm, illuminated with a red laser beam (wavelength 630nm) and distance to camera of 250mm.*

The first step to evaluate the Neumann εERC (4.8) is to calculate the correlation $|\langle Ke_i, Ke_j\rangle|$. In the following we assume that all particles are located in a plane parallel to the detector, i.e. $z := z_i$ is constant for all i. Define the function $C : \mathbb{R}^+ \to \mathbb{R}$ for $d \in \mathbb{R}$ by

$$C(d) := \begin{cases} 2r^2 \cos^{-1}\left(\frac{d}{2r}\right) - \frac{d}{2}\sqrt{4r^2 - d^2}, & \text{for } 4r^2 > d^2, \\ 0, & \text{else.} \end{cases}$$

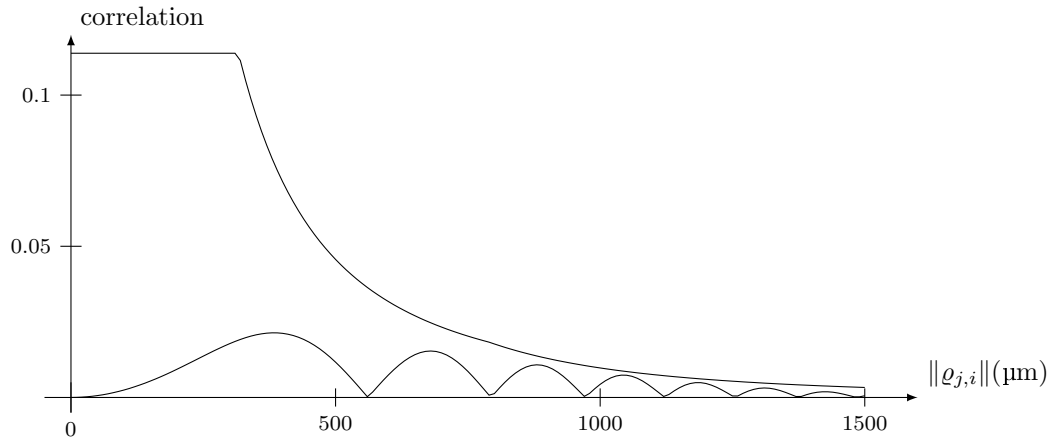


FIG. 5.2. *The oscillating part of the correlation of two atoms with distance* $\|\varrho_{j,i}\|$ *and its corresponding monotonically decreasing estimate (same settings as in figure 5.1).*

In [9] it has been shown that with this definition the correlation $|\langle Ke_i, Ke_j\rangle|$ with distance $\varrho_{j,i} := (x_j - x_i, y_j - y_i)$ in digital holography emerges as

$$|\langle Ke_i, Ke_j\rangle|(\varrho_{j,i}) \approx \frac{C(\|\varrho_{j,i}\|)}{\pi r^2} + \frac{1}{4}\, J_1^2\Big(\frac{2\pi r}{\lambda z}\|\varrho_{j,i}\|\Big)\, \Big|\operatorname{sinc}\Big(\frac{1}{2\lambda z}\|\varrho_{j,i}\|^2\Big)\Big|, \qquad (5.3)$$

where sinc denotes the normalized sine cardinal function and is defined via $\operatorname{sinc}(x) := \sin(\pi x)/\pi x$.

The correlation in digital holography (5.3) is challengingly usable, because it is not monotonically decreasing in the distance $\|\varrho_{j,i}\| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$ due to the oscillating Bessel and sine functions. To come to an estimate from above, which is monotonically decreasing, we use bounds for the absolute value of the Bessel function. With known constants $b_L \approx 0.6748$ and $c_L \approx 0.7857$ the correlation (5.3) can be estimated by the following majorizing function $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}$ [9]

$$\begin{aligned}|\langle Ke_i, Ke_j\rangle|(\varrho_{j,i}) &\le \mathcal{M}(\|\varrho_{j,i}\|)\\ &:= \frac{C(\|\varrho_{j,i}\|}{\pi r^2} + \frac{1}{4}\min\Big\{b_L^2, c_L^2\Big(\frac{\lambda z}{2\pi r}\Big)^{\frac{2}{3}}\|\varrho_{j,i}\|^{-\frac{2}{3}}\Big\}\min\Big\{1, \frac{2\lambda z}{\pi}\|\varrho_{j,i}\|^{-2}\Big\}, \qquad (5.4)\end{aligned}$$

which is monotonically decreasing in $\|\varrho_{j,i}\|$. Figure 5.2 illustrates the oscillating part of the correlation (5.3) and its corresponding upper bound for two particles of 50µm radius which are illuminated with a red laser beam (wavelength 630nm) in a distance of 250mm to the camera.

With the estimate (5.4), we come to a resolution bound for droplets jet reconstruction, as e.g. used in [22]. Here monodisperse droplets (i.e. they have the same size, shape and mass) were generated and emitted on a strait line parallel to the detector plane. This configuration eases the computation of the εNeumann ERC. We define that the particles are located at some grid points

$$\Delta\mathbb{Z} := \{i \in \mathbb{Z} \,|\, i/\Delta \in \mathbb{Z}\},$$

where the parameter Δ describes the grid refinement. Assume that the particles have the minimal distance

$$\rho := \min_{i,j \in \operatorname{supp}(u^\diamond)} \|\varrho_{j,i}\| \in \Delta\mathbb{N},$$

then the sums of correlations COR_I and $\mathrm{COR}_{I^\complement}$ can be estimated from above. W.l.o.g. we fix one particle at the origin and estimate with the worst case that the other particles appear at a distance of $j\rho$ to the origin, with $-\lfloor N/2\rfloor \leq j \leq \lfloor N/2\rfloor$. Then, for $\rho > \Delta$ we get

$$\mathrm{COR}_I = \sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle Ke_i, Ke_j\rangle| \leq 2 \sum_{j=1}^{\lfloor N/2\rfloor} \mathcal{M}(j\rho),$$

$$\mathrm{COR}_{I^\complement} = \sup_{i\in I^\complement} \sum_{j\in I} |\langle Ke_i, Ke_j\rangle| \leq \sup_{\substack{i\in\Delta\mathbb{Z}\\ \Delta\leq i\leq \rho-\Delta}} \sum_{j=-\lfloor N/2\rfloor}^{\lfloor N/2\rfloor} \mathcal{M}(|j\rho - i|).$$

Remark 7. If the cardinality N of the support I is unknown then one can replace the finite sums by infinite sums. These sums exist and can be expressed in terms of the Hurwitz zeta function $\zeta(\nu, q) := \sum_{j=0}^{\infty}(q+j)^{-\nu}$, for $\nu > 1$ and $q > 0$, and the Riemann zeta function $\zeta(\nu) := \zeta(\nu, 1) = \sum_{j=1}^{\infty} j^{-\nu}$, respectively.

Consequently, we can formulate an estimate for the Neumann εERC (4.8).

Proposition 5.1 (Neumann εERC for Fresnel-convolved characteristic functions). *An estimate from above for the Neumann εERC (4.8) for characteristic functions convolved with the real part of the Fresnel kernel is for $\rho > \Delta$*

$$2 \sum_{j=1}^{\lfloor N/2\rfloor} \mathcal{M}(j\rho) + \sup_{1\leq i< \frac{\rho}{\Delta}} \sum_{j=-\lfloor N/2\rfloor}^{\lfloor N/2\rfloor} \mathcal{M}(|j\rho - i\Delta|) < 1 - 2r_{\varepsilon/u}. \tag{5.5}$$

This means, that there is a regularization parameter α which allows exact recovery of the support with the ℓ^1-penalized Tikhonov regularization, if the above condition is fulfilled.

Remark 8. In comparison, the exact recovery condition in terms of the coherence parameter $\mu = \sup_{i\neq j} |\langle Ke_i, Ke_j\rangle|(\varrho_{j,i}) \leq \sup_{i\neq j} \mathcal{M}(\|\varrho_{j,i}\|) = \mathcal{M}(\Delta)$ according to Fuchs' exact recovery condition from [14] appears as

$$(2N-1)\,\mathcal{M}(\Delta) < 1 - 2r_{\varepsilon/u},$$

see Remark 6. This condition is significantly worse than (5.5), since asymptotically it holds that $\mathcal{M}(\|\varrho_{j,i}\|) \sim \|\varrho_{j,i}\|^{-\frac{8}{3}}$.

Condition (5.5) of Proposition 5.1 seems not to be easy to handle due to the upper bound $\mathcal{M}$ from (5.4). However, in practice all parameters are known, and one can compute a bound via approaching from large ρ. As soon as the sum is smaller than $1 - 2r_{\varepsilon/u}$, it is guaranteed that the ℓ^1-penalized Tikhonov regularization can recover exactly. A typical setting for digital holography of particles is the usage of a red laser of wavelength $\lambda = 0.6328$µm and a distance of $z = 200$mm from the camera, cf. [22]. In figure 5.3 condition (5.5) of Proposition 5.1 is plotted for particles with typical radii of $r \in \{5, 25, 50, 100\}$µm. In the computation the asymptotic formula is used, i.e. for an unknown support cardinality N. For the dictionaries a corresponding refinement of $\Delta = r/2$ was chosen. The colored areas describe the combinations where the Neumann εERC (5.5) is fulfilled, and hence exact recovery is possible.

We apply the Neumann εERC (5.5) to simulated data of droplets jets. For the simulation we use the same setting as above, i.e. a red laser of wavelength $\lambda = 0.6328$µm and a distance of $z = 200$mm from the camera. The particles have a diameter of

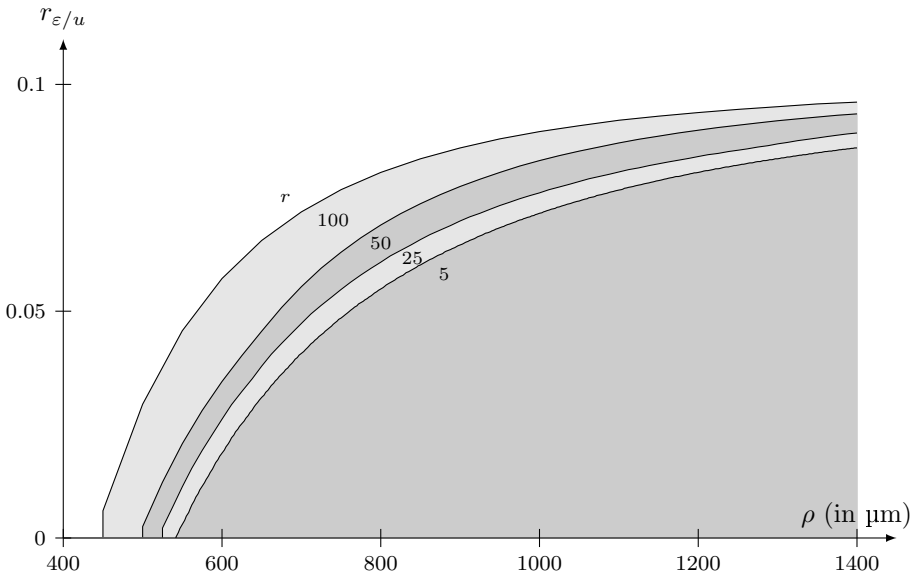


FIG. 5.3. *The Neumann εERC (5.5) for combinations of ρ and $r_{\varepsilon/u}$ with the asymptotic formula for an unknown support cardinality N. For particles the radii $r \in \{5, 25, 50, 100\}\mu m$ were used, and for the corresponding grid we chose a refinement of $\Delta = r/2$.*

100µm and for the corresponding grid we choose a refinement of 25µm. Those parameters correspond to that of the experimental setup used in [22, 23].

After applying the digital holography model (5.1), we add Gaussian noise of different noise levels and in each case of zero mean. For the coefficients $u_i^\diamond$, we choose a setting which implies $u_i^\diamond \approx 10$ for all $i \in I$. Figure 5.4 shows three simulated holograms with different distances ρ and different noise-to-signal ratios $r_{\varepsilon/u}$. For all three noisy examples in the right column all the particles were recovered exactly. For minimization of the Tikhonov functional we used the iterated soft-thresholding algorithm [7]. However, only for the image on top ($\rho \approx 721$µm) condition (5.5) of Proposition 5.1 holds. In the image in the middle of figure 5.4, the particles have a too small distance to each other ($\rho \approx 360$µm), and even for the noiseless case condition (5.5) is not fulfilled. The last image ($\rho \approx 721$µm) was manipulated with unrealistically huge noise, so that condition (5.5) is violated, too.

6. Conclusion. With the papers [15] and [16], the analysis of a priori parameter rules for ℓ^1-penalized Tikhonov functionals seemed completed. On the common parameter rule $\alpha \asymp \varepsilon$, linear, i.e. best possible, convergence is guaranteed. In this paper we have gone beyond this question by presenting a parameter rule which ensures exact recovery of the unknown support of $u^\diamond \in \ell^0$. Moreover, on that condition we achieve a linear convergence rate measured in the ℓ^1 norm, that comes with a-priori checkable error constants which are easier to handle than the ones from [15]. A side product of our analysis is the proof of convergence in ℓ^0 in the topology of sparse convergence.

Granted, to apply the Neumann εERC (4.8) and the Neumann parameter rule (4.9) one has to know the support I. However, with a certain prior knowledge the correla-

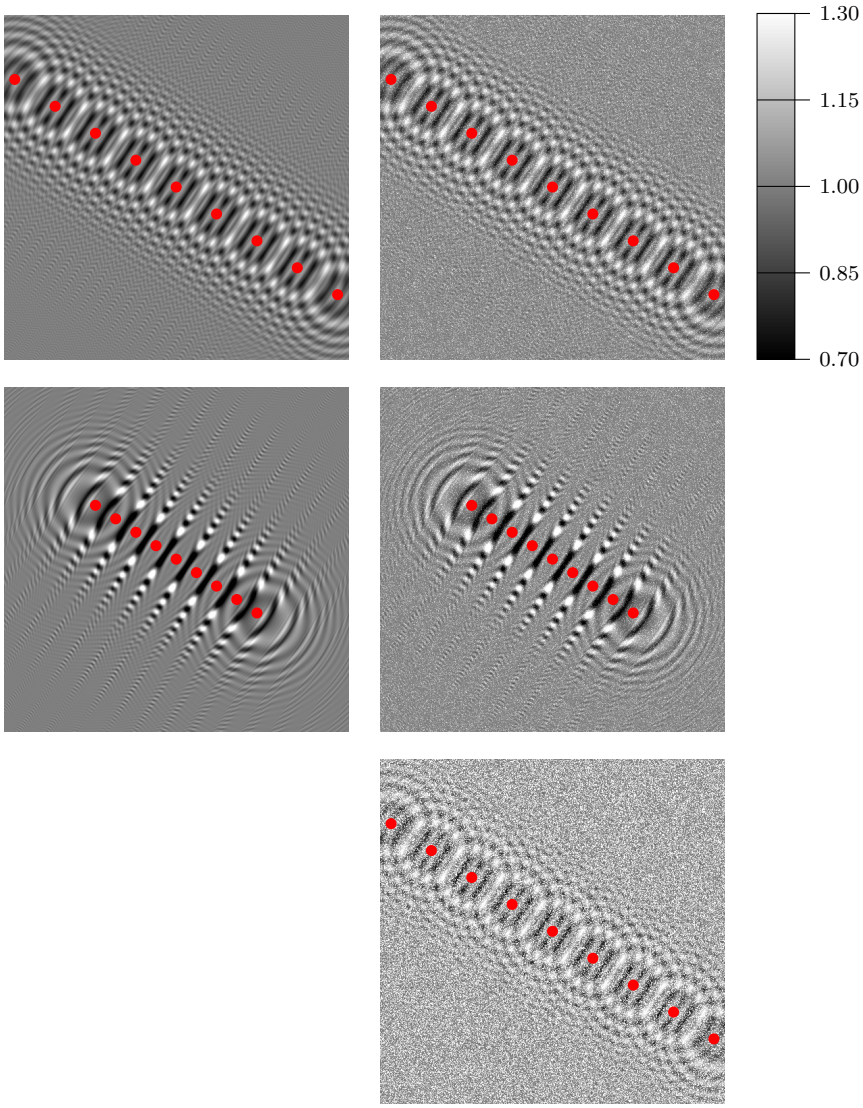


FIG. 5.4. *Simulated holograms of spherical particles. In the left column the noiseless signals are displayed. For reconstruction, the noisy signals of the right column are used. The dots correspond to the true location of the particles. The ℓ^1-penalized Tikhonov regularization recovered all particles exactly, however, condition (5.5) of Proposition 5.1 was just fulfilled for the image on top. In the image in the middle the particles have a too small distance to each other, and at the bottom the image was manipulated with unrealistically huge noise.*

tions

$$\mathrm{COR}_I := \sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle Ke_i, Ke_j\rangle| \quad \text{and} \quad \mathrm{COR}_{I^\complement} := \sup_{i\in I^\complement} \sum_{j\in I} |\langle Ke_i, Ke_j\rangle|,$$

can be estimated from above a priori, especially when the support I is not known exactly. That way it is possible to obtain a priori computable conditions for exact recovery. In §5 it has be done exemplarily for characteristic functions convolved with

a Fresnel function. This shows the practical relevance of the condition.

REFERENCES

[1] Kristian Bredies and Dirk A. Lorenz, *Linear convergence of iterative soft-thresholding*, Journal of Fourier Analysis and Applications, 14 (2008), pp. 813–837.

[2] Emmanuel J. Candés and Justin K. Romberg, *Sparsity and incoherence in compressive sampling*, Inverse Problems, 23 (2007), pp. 969–985.

[3] Emmanuel J. Candés, Justin K. Romberg, and Terence Tao, *Stable signal recovery from incomplete and inaccurate measurements*, Communications on Pure and Applied Mathematics, 59 (2006), pp. 1207–1223.

[4] Emmanuel J. Candés and Terence Tao, *Decoding by linear programming*, IEEE Transaction on Information Theory, 51 (2005), pp. 4203–4215.

[5] Scott Shaobing Chen, David L. Donoho, and Michael A. Saunders, *Atomic decomposition by basis pursuit*, SIAM Journal on Scientific Computing, 20 (1998), pp. 33–61.

[6] Stephan Dahlke, Massimo Fornasier, and Thorsten Raasch, *Multilevel preconditioning for adaptive sparse optimization*. Preprint 25, DFG SPP 1324, 2009.

[7] Ingrid Daubechies, Michel Defrise, and Christine De Mol, *An iterative thresholding algorithm for linear inverse problems with a sparsity constraint*, Communications in Pure and Applied Mathematics, 57 (2004), pp. 1413–1457.

[8] Loïc Denis, Dirk A. Lorenz, Eric Thiébaut, Corinne Fournier, and Dennis Trede, *Inline hologram reconstruction with sparsity constraints*, Optics Letters, 34 (2009), pp. 3475–3477.

[9] Loïc Denis, Dirk A. Lorenz, and Dennis Trede, *Greedy solution of ill-posed problems: Error bounds and exact inversion*, Inverse Problems, 25 (2009), p. 115017 (24pp).

[10] Jean Dieudonné and Laurent Schwartz, *La dualité dans les espaces* $(\mathcal{F})$ *et* $(\mathcal{LF})$, Annales de l'Institute Fourier Grenoble, 1 (1949), pp. 61–101 (1950).

[11] Charles Dossal and Stéphane Mallat, *Sparse spike deconvolution with minimum scale*, in Proceedings of the First Workshop "Signal Processing with Adaptive Sparse Structured Representations", 2005.

[12] Heinz W. Engl, Martin Hanke, and Andreas Neubauer, *Regularization of Inverse Problems*, vol. 375 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 2000.

[13] Jean-Jacques Fuchs, *On sparse representations in arbitrary redundant bases*, IEEE Transactions on Information Theory, 50 (2004), pp. 1341–1344.

[14] ———, *Recovery of exact sparse representations in the presence of bounded noise*, IEEE Transactions on Information Theory, 51 (2005), pp. 3601–3608.

[15] Markus Grasmair, Markus Haltmeier, and Otmar Scherzer, *Sparse regularization with* ℓ^q *penalty term*, Inverse Problems, 24 (2008), p. 055020 (13pp).

[16] Markus Grasmair, Markus Haltmeier, and Otmar Scherzer, *Necessary and sufficient conditions for linear convergence of* ℓ^1*-regularization*. Report 18, FSP S105, 2009.

[17] Rémi Gribonval and Morten Nielsen, *Beyond sparsity: Recovering structured representations by* ℓ^1 *minimization and greedy algorithms*, Advances in Computational Mathematics, 28 (2008), pp. 23–41.

[18] Roland Griesse and Dirk A. Lorenz, *A semismooth Newton method for Tikhonov functionals with sparsity constraints*, Inverse Problems, 24 (2008), p. 035007 (19pp).

[19] Bangti Jin, Dirk A. Lorenz, and Stefan Schiffler, *Elastic-net regularization: Error estimates and active set methods*, Inverse Problems, 25 (2009), p. 115022 (26pp).

[20] Dirk A. Lorenz, *Convergence rates and source conditions for Tikhonov regularization with sparsity constraints*, Journal of Inverse and Ill-Posed Problems, 16 (2008), pp. 463–478.

[21] Lawrence Narici and Edward Beckenstein, *Topological vector spaces*, Marcel Dekker Inc., New York, 1985.

[22] Ferréol Soulez, Loïc Denis, Corinne Fournier, Éric Thiébaut, and Charles Goepfert, *Inverse problem approach for particle digital holography: accurate location based on local optimisation*, Journal of the Optical Society of America A, 24 (2007), pp. 1164–1171.

[23] F. Soulez, L. Denis, É. Thiébaut, C. Fournier, and C. Goepfert, *Inverse problem approach in particle digital holography: out-of-field particle detection made possible*, Journal of the Optical Society of America A, 24 (2007), pp. 3708–3716.

[24] Dennis Trede, *Inverse Problems with Sparsity Constraints: Convergence Rates and Exact Recovery*, PhD thesis, Univerität Bremen, 2010.

[25] Joel A. Tropp, *Just relax: Convex programming methods for identifying sparse signals in noise*, IEEE Transactions on Information Theory, 52 (2006), pp. 1030–1051.

Preprint Series DFG-SPP 1324

http://www.dfg-spp1324.de

Reports

[1] R. Ramlau, G. Teschke, and M. Zhariy. A Compressive Landweber Iteration for Solving Ill-Posed Inverse Problems. Preprint 1, DFG-SPP 1324, September 2008.

[2] G. Plonka. The Easy Path Wavelet Transform: A New Adaptive Wavelet Transform for Sparse Representation of Two-dimensional Data. Preprint 2, DFG-SPP 1324, September 2008.

[3] E. Novak and H. Woźniakowski. Optimal Order of Convergence and (In-) Tractability of Multivariate Approximation of Smooth Functions. Preprint 3, DFG-SPP 1324, October 2008.

[4] M. Espig, L. Grasedyck, and W. Hackbusch. Black Box Low Tensor Rank Approximation Using Fibre-Crosses. Preprint 4, DFG-SPP 1324, October 2008.

[5] T. Bonesky, S. Dahlke, P. Maass, and T. Raasch. Adaptive Wavelet Methods and Sparsity Reconstruction for Inverse Heat Conduction Problems. Preprint 5, DFG-SPP 1324, January 2009.

[6] E. Novak and H. Woźniakowski. Approximation of Infinitely Differentiable Multivariate Functions Is Intractable. Preprint 6, DFG-SPP 1324, January 2009.

[7] J. Ma and G. Plonka. A Review of Curvelets and Recent Applications. Preprint 7, DFG-SPP 1324, February 2009.

[8] L. Denis, D. A. Lorenz, and D. Trede. Greedy Solution of Ill-Posed Problems: Error Bounds and Exact Inversion. Preprint 8, DFG-SPP 1324, April 2009.

[9] U. Friedrich. A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs. Preprint 9, DFG-SPP 1324, April 2009.

[10] K. Bredies and D. A. Lorenz. Minimization of Non-smooth, Non-convex Functionals by Iterative Thresholding. Preprint 10, DFG-SPP 1324, April 2009.

[11] K. Bredies and D. A. Lorenz. Regularization with Non-convex Separable Constraints. Preprint 11, DFG-SPP 1324, April 2009.

[12] M. Döhler, S. Kunis, and D. Potts. Nonequispaced Hyperbolic Cross Fast Fourier Transform. Preprint 12, DFG-SPP 1324, April 2009.

[13] C. Bender. Dual Pricing of Multi-Exercise Options under Volume Constraints. Preprint 13, DFG-SPP 1324, April 2009.

[14] T. Müller-Gronbach and K. Ritter. Variable Subspace Sampling and Multi-level Algorithms. Preprint 14, DFG-SPP 1324, May 2009.

[15] G. Plonka, S. Tenorth, and A. Iske. Optimally Sparse Image Representation by the Easy Path Wavelet Transform. Preprint 15, DFG-SPP 1324, May 2009.

[16] S. Dahlke, E. Novak, and W. Sickel. Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings IV: Errors in L_2 and Other Norms. Preprint 16, DFG-SPP 1324, June 2009.

[17] B. Jin, T. Khan, P. Maass, and M. Pidcock. Function Spaces and Optimal Currents in Impedance Tomography. Preprint 17, DFG-SPP 1324, June 2009.

[18] G. Plonka and J. Ma. Curvelet-Wavelet Regularized Split Bregman Iteration for Compressed Sensing. Preprint 18, DFG-SPP 1324, June 2009.

[19] G. Teschke and C. Borries. Accelerated Projected Steepest Descent Method for Nonlinear Inverse Problems with Sparsity Constraints. Preprint 19, DFG-SPP 1324, July 2009.

[20] L. Grasedyck. Hierarchical Singular Value Decomposition of Tensors. Preprint 20, DFG-SPP 1324, July 2009.

[21] D. Rudolf. Error Bounds for Computing the Expectation by Markov Chain Monte Carlo. Preprint 21, DFG-SPP 1324, July 2009.

[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.

[23] F.J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter. Multi-level Monte Carlo Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 23, DFG-SPP 1324, August 2009.

[24] S. Dereich and F. Heidenreich. A Multilevel Monte Carlo Algorithm for Lévy Driven Stochastic Differential Equations. Preprint 24, DFG-SPP 1324, August 2009.

[25] S. Dahlke, M. Fornasier, and T. Raasch. Multilevel Preconditioning for Adaptive Sparse Optimization. Preprint 25, DFG-SPP 1324, August 2009.

[26] S. Dereich. Multilevel Monte Carlo Algorithms for Lévy-driven SDEs with Gaussian Correction. Preprint 26, DFG-SPP 1324, August 2009.

[27] G. Plonka, S. Tenorth, and D. Roşca. A New Hybrid Method for Image Approximation using the Easy Path Wavelet Transform. Preprint 27, DFG-SPP 1324, October 2009.

[28] O. Koch and C. Lubich. Dynamical Low-rank Approximation of Tensors. Preprint 28, DFG-SPP 1324, November 2009.

[29] E. Faou, V. Gradinaru, and C. Lubich. Computing Semi-classical Quantum Dynamics with Hagedorn Wavepackets. Preprint 29, DFG-SPP 1324, November 2009.

[30] D. Conte and C. Lubich. An Error Analysis of the Multi-configuration Time-dependent Hartree Method of Quantum Dynamics. Preprint 30, DFG-SPP 1324, November 2009.

[31] C. E. Powell and E. Ullmann. Preconditioning Stochastic Galerkin Saddle Point Problems. Preprint 31, DFG-SPP 1324, November 2009.

[32] O. G. Ernst and E. Ullmann. Stochastic Galerkin Matrices. Preprint 32, DFG-SPP 1324, November 2009.

[33] F. Lindner and R. L. Schilling. Weak Order for the Discretization of the Stochastic Heat Equation Driven by Impulsive Noise. Preprint 33, DFG-SPP 1324, November 2009.

[34] L. Kämmerer and S. Kunis. On the Stability of the Hyperbolic Cross Discrete Fourier Transform. Preprint 34, DFG-SPP 1324, December 2009.

[35] P. Cerejeiras, M. Ferreira, U. Kähler, and G. Teschke. Inversion of the noisy Radon transform on $SO(3)$ by Gabor frames and sparse recovery principles. Preprint 35, DFG-SPP 1324, January 2010.

[36] T. Jahnke and T. Udrescu. Solving Chemical Master Equations by Adaptive Wavelet Compression. Preprint 36, DFG-SPP 1324, January 2010.

[37] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Irregular Shearlet Frames: Geometry and Approximation Properties. Preprint 37, DFG-SPP 1324, February 2010.

[38] G. Kutyniok and W.-Q Lim. Compactly Supported Shearlets are Optimally Sparse. Preprint 38, DFG-SPP 1324, February 2010.

[39] M. Hansen and W. Sickel. Best m-Term Approximation and Tensor Products of Sobolev and Besov Spaces – the Case of Non-compact Embeddings. Preprint 39, DFG-SPP 1324, March 2010.

[40] B. Niu, F.J. Hickernell, T. Müller-Gronbach, and K. Ritter. Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 40, DFG-SPP 1324, March 2010.

[41] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Construction of Compactly Supported Shearlet Frames. Preprint 41, DFG-SPP 1324, March 2010.

[42] C. Bender and J. Steiner. Error Criteria for Numerical Solutions of Backward SDEs. Preprint 42, DFG-SPP 1324, April 2010.

[43] L. Grasedyck. Polynomial Approximation in Hierarchical Tucker Format by Vector-Tensorization. Preprint 43, DFG-SPP 1324, April 2010.

[44] M. Hansen und W. Sickel. Best m-Term Approximation and Sobolev-Besov Spaces of Dominating Mixed Smoothness - the Case of Compact Embeddings. Preprint 44, DFG-SPP 1324, April 2010.

[45] P. Binev, W. Dahmen, and P. Lamby. Fast High-Dimensional Approximation with Sparse Occupancy Trees. Preprint 45, DFG-SPP 1324, May 2010.

[46] J. Ballani and L. Grasedyck. A Projection Method to Solve Linear Systems in Tensor Format. Preprint 46, DFG-SPP 1324, May 2010.

[47] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence Rates for Greedy Algorithms in Reduced Basis Methods. Preprint 47, DFG-SPP 1324, May 2010.

[48] S. Kestler and K. Urban. Adaptive Wavelet Methods on Unbounded Domains. Preprint 48, DFG-SPP 1324, June 2010.

[49] H. Yserentant. The Mixed Regularity of Electronic Wave Functions Multiplied by Explicit Correlation Factors. Preprint 49, DFG-SPP 1324, June 2010.

[50] H. Yserentant. On the Complexity of the Electronic Schrödinger Equation. Preprint 50, DFG-SPP 1324, June 2010.

[51] M. Guillemard and A. Iske. Curvature Analysis of Frequency Modulated Manifolds in Dimensionality Reduction. Preprint 51, DFG-SPP 1324, June 2010.

[52] E. Herrholz and G. Teschke. Compressive Sensing Principles and Iterative Sparse Recovery for Inverse and Ill-Posed Problems. Preprint 52, DFG-SPP 1324, July 2010.

[53] L. Kämmerer, S. Kunis, and D. Potts. Interpolation Lattices for Hyperbolic Cross Trigonometric Polynomials. Preprint 53, DFG-SPP 1324, July 2010.

[54] G. Kutyniok and W.-Q Lim. Shearlets on Bounded Domains. Preprint 54, DFG-SPP 1324, July 2010.

[55] A. Zeiser. Wavelet Approximation in Weighted Sobolev Spaces of Mixed Order with Applications to the Electronic Schrödinger Equation. Preprint 55, DFG-SPP 1324, July 2010.

[56] G. Kutyniok, J. Lemvig, and W.-Q Lim. Compactly Supported Shearlets. Preprint 56, DFG-SPP 1324, July 2010.

[57] A. Zeiser. On the Optimality of the Inexact Inverse Iteration Coupled with Adaptive Finite Element Methods. Preprint 57, DFG-SPP 1324, July 2010.

[58] S. Jokar. Sparse Recovery and Kronecker Products. Preprint 58, DFG-SPP 1324, August 2010.

[59] T. Aboiyar, E. H. Georgoulis, and A. Iske. Adaptive ADER Methods Using Kernel-Based Polyharmonic Spline WENO Reconstruction. Preprint 59, DFG-SPP 1324, August 2010.

[60] O. G. Ernst, A. Mugler, H.-J. Starkloff, and E. Ullmann. On the Convergence of Generalized Polynomial Chaos Expansions. Preprint 60, DFG-SPP 1324, August 2010.

[61] S. Holtz, T. Rohwedder, and R. Schneider. On Manifolds of Tensors of Fixed TT-Rank. Preprint 61, DFG-SPP 1324, September 2010.

[62] J. Ballani, L. Grasedyck, and M. Kluge. Black Box Approximation of Tensors in Hierarchical Tucker Format. Preprint 62, DFG-SPP 1324, October 2010.

[63] M. Hansen. On Tensor Products of Quasi-Banach Spaces. Preprint 63, DFG-SPP 1324, October 2010.

[64] S. Dahlke, G. Steidl, and G. Teschke. Shearlet Coorbit Spaces: Compactly Supported Analyzing Shearlets, Traces and Embeddings. Preprint 64, DFG-SPP 1324, October 2010.

[65] W. Hackbusch. Tensorisation of Vectors and their Efficient Convolution. Preprint 65, DFG-SPP 1324, November 2010.

[66] P. A. Cioica, S. Dahlke, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Spatial Besov Regularity for Stochastic Partial Differential Equations on Lipschitz Domains. Preprint 66, DFG-SPP 1324, November 2010.

[67] E. Novak and H. Woźniakowski. On the Power of Function Values for the Approximation Problem in Various Settings. Preprint 67, DFG-SPP 1324, November 2010.

[68] A. Hinrichs, E. Novak, and H. Woźniakowski. The Curse of Dimensionality for Monotone and Convex Functions of Many Variables. Preprint 68, DFG-SPP 1324, November 2010.

[69] G. Kutyniok and W.-Q Lim. Image Separation Using Shearlets. Preprint 69, DFG-SPP 1324, November 2010.

[70] B. Jin and P. Maass. An Analysis of Electrical Impedance Tomography with Applications to Tikhonov Regularization. Preprint 70, DFG-SPP 1324, December 2010.

[71] S. Holtz, T. Rohwedder, and R. Schneider. The Alternating Linear Scheme for Tensor Optimisation in the TT Format. Preprint 71, DFG-SPP 1324, December 2010.

[72] T. Müller-Gronbach and K. Ritter. A Local Refinement Strategy for Constructive Quantization of Scalar SDEs. Preprint 72, DFG-SPP 1324, December 2010.

[73] T. Rohwedder and R. Schneider. An Analysis for the DIIS Acceleration Method used in Quantum Chemistry Calculations. Preprint 73, DFG-SPP 1324, December 2010.

[74] C. Bender and J. Steiner. Least-Squares Monte Carlo for Backward SDEs. Preprint 74, DFG-SPP 1324, December 2010.

[75] C. Bender. Primal and Dual Pricing of Multiple Exercise Options in Continuous Time. Preprint 75, DFG-SPP 1324, December 2010.

[76] H. Harbrecht, M. Peters, and R. Schneider. On the Low-rank Approximation by the Pivoted Cholesky Decomposition. Preprint 76, DFG-SPP 1324, December 2010.

[77] P. A. Cioica, S. Dahlke, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Adaptive Wavelet Methods for Elliptic Stochastic Partial Differential Equations. Preprint 77, DFG-SPP 1324, January 2011.

[78] G. Plonka, S. Tenorth, and A. Iske. Optimal Representation of Piecewise Hölder Smooth Bivariate Functions by the Easy Path Wavelet Transform. Preprint 78, DFG-SPP 1324, January 2011.

[79] A. Mugler and H.-J. Starkloff. On Elliptic Partial Differential Equations with Random Coefficients. Preprint 79, DFG-SPP 1324, January 2011.

[80] T. Müller-Gronbach, K. Ritter, and L. Yaroslavtseva. A Derandomization of the Euler Scheme for Scalar Stochastic Differential Equations. Preprint 80, DFG-SPP 1324, January 2011.

[81] W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive Petrov-Galerkin methods for first order transport equations. Preprint 81, DFG-SPP 1324, January 2011.

[82] K. Grella and C. Schwab. Sparse Tensor Spherical Harmonics Approximation in Radiative Transfer. Preprint 82, DFG-SPP 1324, January 2011.

[83] D.A. Lorenz, S. Schiffler, and D. Trede. Beyond Convergence Rates: Exact Inversion With Tikhonov Regularization With Sparsity Constraints. Preprint 83, DFG-SPP 1324, January 2011.