DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Complexity of Oscillatory Integration for Univariate Sobolev Spaces

E. Novak, M. Ullrich, H. Woźniakowski

Preprint 151

Edited by

AG Numerik/Optimierung
Fachbereich 12 - Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Str.
35032 Marburg

DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Complexity of Oscillatory Integration for Univariate Sobolev Spaces

E. Novak, M. Ullrich, H. Woźniakowski

Preprint 151

Complexity of Oscillatory Integration for Univariate Sobolev Spaces

Erich Novak*, Mario Ullrich†
Mathematisches Institut, Universität Jena
Ernst-Abbe-Platz 2, 07743 Jena, Germany
email: erich.novak@uni-jena.de, ullrich.mario@gmail.com
Henryk Woźniakowski‡
Department of Computer Science, Columbia University,
New York, NY 10027, USA, and
Institute of Applied Mathematics, University of Warsaw
ul. Banacha 2, 02-097 Warszawa, Poland
email: henryk@cs.columbia.edu

November 6, 2013

Abstract

We analyze univariate oscillatory integrals for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. We find matching lower and upper bounds on the minimal worst case error of algorithms that use n function or derivative values. We also find sharp bounds on the information complexity which is the minimal n for which the absolute or normalized error is at most ε. We show surprising relations between the information complexity and the oscillatory weight. We also briefly consider the case of $s = \infty$.

*This author was partially supported by the DFG-Priority Program 1324.
†This author was partially supported by the DFG GRK 1523.
‡This author was partially supported by the National Science Foundation.

1 Introduction

There are many recent papers about the approximate computation of highly oscillatory univariate integrals with the weight $\exp(2\pi\, i\, kx)$, where $x \in [0,1]$ and k is an integer which is assumed to be large in the absolute sense, see Domínguez, Graham and Smyshlyaev [4], Iserles and Nørsett [6], Melenk [8], Chapter 3 of Olver [11], and Huybrechs and Olver [5] for a survey. Many papers present asymptotic error bounds as k goes to infinity for algorithms that use n function or derivative values. It is usually done for C^∞ or even analytic functions. There are not too many papers that contain explicit error bounds depending on k and n. Examples include [4, 8, 11]. All these papers also contain pointers to the further relevant literature.

There seems to be some doubt in the literature concerning the question whether "high oscillation", i.e., large $|k|$, means that the problem is "easy" or "difficult". We believe that this uncertainty is related to the fact that error bounds are rare and most authors do not distinguish between the absolute and normalized error criteria. The absolute error criteria means that the error is at most ε, whereas the normalized error criteria means that the error is at most ε times the initial error. The initial error is the error of the zero algorithm and only depends on the formulation of the problem. It turns out that in the setting of our paper the initial error is small for large $|k|$ which makes the absolute error criterion easier than the normalized error criterion. We show that the answer to the question whether the problem is easy or difficult for large $|k|$ depends on the error criterion we choose as well on the relation between $|k|$, ε and the assumed smoothness of integrands.

We did not find a computation of the initial error in the literature and we did not find lower bounds on the error of algorithms that use n function or derivatives values. In this paper, we present the formulas for the initial error as well as matching lower and upper bounds on the minimal errors of algorithms.

More precisely, we study the approximate computation of univariate oscillatory integrals for the standard Sobolev spaces H^s of periodic and non-periodic functions defined on $[0,1]$ with an arbitrary integer $s \geq 1$. We usually consider a finite s but we also briefly consider the case of $s = \infty$. Although we consider arbitrary integers k, our emphasis is for large $|k|$ and we explain our results here only for such k.

For a finite s, we obtain matching lower and upper bounds on the nth minimal worst case errors of algorithms that use n function or derivatives values. For $n = 0$, it is the initial error. For the periodic case the initial error is of order $|k|^{-s}$, whereas for the non-periodic case it is independent of s and is roughly $|k|^{-1}$. This means that the initial error for the periodic case is much smaller for large s.

For $s = \infty$, the periodic case leads to the space of only constant functions and the problem becomes trivial since the initial error is zero for all $k \neq 0$. The non-periodic case is still reasonable with the initial error roughly $|k|^{-1}$.

For a finite s and the periodic case, we prove that an algorithm that uses n function values at equally spaced points is nearly optimal and its worst case error is bounded by $C_s(n+|k|)^{-s}$ with an exponentially small C_s in s. For the non-periodic case, we first compute successive derivatives up to order $s-1$ at the end-points $x=0$ and $x=1$. These derivatives values are used to periodize the function and this allows us to obtain similar error bounds like for the periodic case. Asymptotically in n, the worst case error of the algorithm is of order n^{-s} independently of k for both periodic and non-periodic cases.

Near optimality of this algorithm is shown by proving a lower bound of order $(n+|k|)^{-s}$ which holds for all algorithms that use the values of function and derivatives up to order $s-1$ at n arbitrarily chosen points from $[0,1]$. We establish the lower bound by constructing a periodic function that vanishes with all its derivatives up to order $s-1$ at the points sampled by a given algorithm, belongs to the unit ball of the space H^s and its oscillatory integral is of order $(n+|k|)^{-s}$.

For $s=\infty$, we provide two algorithms which compute successive derivatives and/or function values at equally spaced points. The worst case error of one of these algorithms is super exponentially small in n. For $s=\infty$, we do not have a matching lower bound.

We consider the absolute and normalized error criteria. For the absolute error criterion, we want to find the information complexity which is defined as the smallest n for which the nth minimal error is at most $\varepsilon \in (0,1)$, whereas for the normalized error criterion, the information complexity is the smallest n for which the nth minimal error reduces the initial error by a factor ε. For a finite s we obtain the following results.

- For the absolute error criterion and the periodic case, the information complexity is zero if $\varepsilon > 1/(2\pi|k|)^s$ and otherwise is roughly $\varepsilon^{-1/s}-|k|$. This means that in this case the problem becomes easier for large $|k|$.

- For the normalized error criterion and for the periodic case, the information complexity is of order $|k|\,\varepsilon^{-1/s}$. Hence, in this case the problem becomes harder for larger $|k|$.

- For the absolute error criterion and the non-periodic case, the information complexity is zero if $\varepsilon \ge 1.026/(2\pi|k|)$ and otherwise is roughly lower bounded by $\varepsilon^{-1/s}-|k|$ and upper bounded by $\varepsilon^{-1/s}+2s-1-|k|$. As for the periodic case, the problem becomes easier for large k.

- For the normalized error criterion and the non-periodic case, the information complexity is of order $|k|^{1/s}\,\varepsilon^{-1/s}$ for very small ε. In this case, the dependence on $|k|$ is more lenient than for the periodic case if s is large.

The dependence on $|k|$ is quite intriguing if $|k|$ goes to infinity. For $s = 1$ and fixed ε, the information complexity goes to infinity linearly with $|k|$. However, the situation is quite different for $s \geq 2$. Then for large $|k|$ the information complexity is bounded by $2s$ if ε is fixed or if ε tends to zero like $|k|^{-\eta}$ with $\eta \in (0, s-1)$.

For $s = \infty$, we obtain only upper bounds on the information complexity. For ε tending to zero they are roughly $\ln(\varepsilon^{-1})/\ln(\ln(\varepsilon^{-1}))$ independently of $|k|$.

2 Preliminaries

We study the Sobolev space H^s for a finite $s \in \mathbb{N}$, i.e.,

$$H^s = \{f : [0,1] \to \mathbb{C} \mid f^{(s-1)} \text{ is abs. cont.}, f^{(s)} \in L_2\} \tag{1}$$

with the inner product

$$\begin{aligned} \langle f, g \rangle_s &= \sum_{\ell=0}^{s-1} \int_0^1 f^{(\ell)}(x)\,\mathrm{d}x \int_0^1 \overline{g^{(\ell)}(x)}\,\mathrm{d}x + \int_0^1 f^{(s)}(x)\,\overline{g^{(s)}(x)}\,\mathrm{d}x \\ &= \sum_{\ell=0}^{s-1} \langle f^{(\ell)}, 1 \rangle_0 \,\overline{\langle g^{(\ell)}, 1 \rangle_0} + \langle f^{(s)}, g^{(s)} \rangle_0 \,, \end{aligned} \tag{2}$$

where $\langle f, g \rangle_0 = \int_0^1 f(x)\,\overline{g(x)}\,\mathrm{d}x$, and norm $\|f\|_{H^s} = \langle f, f \rangle_s^{1/2}$. We later comment on the space H^∞ for $s = \infty$.

Remark 1. Probably the most standard inner product on the Sobolev space H^s is

$$\langle f, g \rangle_{s,*} = \sum_{\ell=0}^{s} \langle f^{(\ell)}, g^{(\ell)} \rangle_0 \,. \tag{3}$$

Obviously, the norms $\|\cdot\|_{H^s}$ and $\|\cdot\|_{H^s_*} = \langle f, f \rangle_{s,*}^{1/2}$ are equivalent. What is more surprising, the bounds on the embedding constants are independent of s and close to one. More precisely, we have

$$\tfrac{12}{13}\,\|f\|_{H^s_*} \leq \|f\|_{H^s} \leq \|f\|_{H^s_*} \qquad \text{for all} \quad f \in H^s \quad \text{and} \quad s \in \mathbb{N}. \tag{4}$$

The second inequality is trivial, whereas the first inequality seems to be new and its proof is given in the appendix.

From (4) it clearly follows that all results presented in this paper for the space H^s equipped with $\langle \cdot, \cdot \rangle_s$ are practically the same as for the space H^s equipped with $\langle \cdot, \cdot \rangle_{s,*}$. We choose to work with the inner product $\langle \cdot, \cdot \rangle_s$ since the analysis in this case is easier and more straightforward. □

We want to solve the following problem:

- What is the complexity of the approximate computation of oscillatory integrals of the form
$$I_k(f) = \int_0^1 f(x)\,\mathrm{e}^{-2\pi\, i\, kx}\,\mathrm{d}x, \quad i = \sqrt{-1}, \tag{5}$$
where $k \in \mathbb{Z}$ and $f \in H^s$? Our emphasize is on large $|k|$. We improve the known upper bounds and also prove matching lower bounds.

We ask (and answer) the same question also for the periodic case, i.e., for the subspace of H^s given by
$$\widetilde{H}^s = \{ f \in H^s \mid f^{(\ell)}(0) = f^{(\ell)}(1) \ \text{ for } \ell = 0, 1, \ldots, s-1 \} \tag{6}$$
equipped with the same inner product as for the space H^s. Note that for $f, g \in \widetilde{H}^s$ this inner product simplifies to
$$\langle f, g \rangle_s \;=\; \langle f, 1 \rangle_0\, \overline{\langle g, 1 \rangle_0} + \left\langle f^{(s)}, g^{(s)} \right\rangle_0 .$$

The results are presented in the following order. We first consider the integration problem for periodic functions, i.e., for functions from $\widetilde{H}^s$, and then, using this knowledge, we analyze the integration problem for the space H^s.

The results of this paper could be stated also for real-valued functions, where $I_k(f)$ can be written, for example, as
$$I_k(f) = \int_0^1 f(x)\cos(2\pi kx)\,\mathrm{d}x$$
for $k \in \mathbb{Z}$, loosing only some negligible constants. We decided to work with complex-valued functions to ease the notation.

3 The periodic case

As already indicated, we first analyze oscillatory integration over $\widetilde{H}^s$. That is, we want to approximate the integral
$$I_k(f) := \int_0^1 f(x)\,\mathrm{e}^{-2\pi\, i\, kx}\,\mathrm{d}x = \int_0^1 f(x)\,\cos(2\pi\, k\, x)\,\mathrm{d}x \;-\; i\int_0^1 f(x)\,\sin(2\pi\, k\, x)\,\mathrm{d}x, \tag{7}$$
where $k \in \mathbb{Z}$ and $f \in \widetilde{H}^s$ with $s \in \mathbb{N}$. Although k can be any integer, the emphasis of this paper is for large $|k|$. In this case the weight functions $\cos(2\pi kx)$ and $\sin(2\pi kx)$ highly

oscillate and therefore the approximation of I_k is called an *(highly) oscillatory integration problem.*

We consider the worst case error on the unit ball of $\widetilde{H}^s$ for algorithms that use function values or, more generally, function and derivatives (up to order $s-1$) values. Note that for $f \in H^s$, the values $f^{(j)}(x)$ are well defined for all $j = 0, 1, \dots, s-1$ and $x \in [0,1]$.

It is well known that adaption does not help, see Bakhvalov [1], and linear algorithms are optimal, see Smolyak [12]. These results can be also found in e.g., [9, 10, 13]. This means that without loss of generality we may consider linear algorithms of the form

$$A_n(f) = \sum_{j=1}^{n} a_j f^{(\ell_j)}(x_j) \tag{8}$$

for some $a_j \in \mathbb{C}$, $\ell_j \in [0, s-1]$ and $x_j \in [0,1]$. Observe that we allow the use of derivatives $f^{(\ell_j)}(x_j)$ as in [6]. An important sub-class is the class of linear algorithms that use only function values, i.e.,

$$A_n(f) = \sum_{j=1}^{n} a_j f(x_j). \tag{9}$$

Of course, for $s = 1$ there is no difference between (8) and (9). We will see that for all s the complexity results are similar for both classes of algorithms (8) and (9).

The worst case error of A_n is defined as

$$\tilde{e}(A_n) = \sup_{f \in \widetilde{H}^s,\, \|f\|_{H^s} \le 1} |I_k(f) - A_n(f)|,$$

whereas the nth minimal worst case error is

$$\tilde{e}(n,k,s) := \inf_{A_n} \tilde{e}(A_n). \tag{10}$$

We use the tilde to indicate that we consider the periodic case, i.e., the class $\widetilde{H}^s$. The particular case $n = 0$ corresponds to the zero algorithm $A_0 = 0$ and gives the so-called *initial error*

$$\tilde{e}(0,k,s) := \sup_{f \in \widetilde{H}^s,\, \|f\|_{H^s} \le 1} |I_k(f)| = \|I_k\|_{\widetilde{H}^s \to \mathbb{C}}. \tag{11}$$

Remark 2. We believe that this is a simple but already interesting model problem for approximating highly oscillatory integrals. Later we plan to study the multivariate case and tractability and we believe that, from a practical point of view, the integrals

$$S_k(f) = \int_{\mathbb{R}^d} f(x)\, \mathrm{e}^{ikx_1} \exp(-\|x\|_2^2)\, \mathrm{d}x$$

for smooth integrands $f : \mathbb{R}^d \to \mathbb{C}$ are more interesting. We start, however, with the integral (5) since it seems to be the simplest interesting case of oscillatory integrals. $\square$

We start with the computation of the initial errors $\tilde{e}(0,k,s) = \|I_k\|_{\widetilde{H}^s \to \mathbb{C}}$. Since I_k is a continuous linear functional defined on the Hilbert space $\widetilde{H}^s$, Riesz's theorem implies that for each $k \in \mathbb{Z}$ and $s \in \mathbb{N}$, there exists a function $\tilde{h}_{k,s} \in \widetilde{H}^s$ such that

$$I_k(f) = \left\langle f, \tilde{h}_{k,s} \right\rangle_s \quad \text{for all} \quad f \in \widetilde{H}^s.$$

The function $\tilde{h}_{k,s}$ is called the representer of I_k for the space $\widetilde{H}^s$. It is well known and easy to show that $\|\tilde{h}_{k,s}\|_{H^s} = \|I_k\|_{\widetilde{H}^s \to \mathbb{C}}$.

To find $\tilde{h}_{k,s}$ consider the particular function $e_k(x) = \mathrm{e}^{2\pi\, i\, kx}$. Clearly, $e_k \in \widetilde{H}^s$. Using integration by parts, we obtain for $k \neq 0$

$$\begin{aligned} \langle f, e_k \rangle_s &= \langle f, 1 \rangle_0 \overline{\langle e_k, 1 \rangle_0} + \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = (-1)^s \left\langle f, e_k^{(2s)} \right\rangle_0 \\ &= (2\pi k)^{2s} \langle f, e_k \rangle_0 = (2\pi k)^{2s}\, I_k(f). \end{aligned} \tag{12}$$

For $k = 0$, we have $\langle f, e_k \rangle_s = \langle f, 1 \rangle_0 = I_0(f)$. Hence we obtain

Proposition 3. Let $k \neq 0$. The representer of I_k is

$$\tilde{h}_{k,s}(x) = (2\pi k)^{-2s}\, \mathrm{e}^{2\pi\, i\, kx}$$

and the initial error is

$$\tilde{e}(0,k,s) = \|\tilde{h}_{k,s}\|_{H^s} = \frac{1}{\big(2\pi |k|\big)^s}.$$

Additionally, for $k = 0$ we have $\tilde{h}_{0,s}(x) = 1$ and $\tilde{e}(0,0,s) = 1$.

We now present a few linear algorithms whose worst case errors are of order $(n + |k|)^{-s}$. We will prove later that this is the best possible order.

For $n \geq 1$, we first define the linear algorithm

$$A_n^{\mathrm{QMC}}(f) = \frac{1}{n} \sum_{j=1}^{n} f\,(j/n)\; \mathrm{e}^{-2\pi\, i\, k\, j/n} \qquad \text{for all} \qquad f \in \widetilde{H}^s.$$

We use the superscript QMC to stress that the algorithm uses equal weights $1/n$ for the function $f(\cdot)\exp(-2\pi\, i\, k \cdot)$. This means that this is a QMC (quasi Monte Carlo) algorithm. As we shall see, the worst case error of A_n^{QMC} is small only if n is sufficiently large with respect to $|k|$. Later, we will modify the algorithm A_n^{QMC} to have a good error bound for all n. First we prove the following theorem.

Theorem 4.

(i) The worst case error of A_n^{QMC}, $n \geq 1$, is

$$\tilde{e}(A_n^{\mathrm{QMC}}) = \left(\sum_{j=1}^{\infty} \left(\frac{1}{\max\{1, (2\pi(jn+k))^{2s}\}} + \frac{1}{\max\{1, (2\pi(jn-k))^{2s}\}} \right) \right)^{1/2}.$$

(ii) For any $1 \leq n \leq |k|$ we have

$$\tilde{e}(A_n^{\mathrm{QMC}}) > \tilde{e}(0, k, s).$$

(iii) For any $n > |k|$ we have

$$\tilde{e}(A_n^{\mathrm{QMC}}) = \left(\sum_{j=1}^{\infty} \left(\frac{1}{(2\pi(jn+k))^{2s}} + \frac{1}{(2\pi(jn-k))^{2s}} \right) \right)^{1/2} \leq \frac{2}{(2\pi)^s} \, \frac{1}{(n-|k|)^s}.$$

(iv) Let $\alpha \in (0,1)$. Then for $n > [(1+\alpha)/(1-\alpha)]\, |k|$ we have

$$\tilde{e}(A_n^{\mathrm{QMC}}) \leq \frac{2}{(2\pi\alpha)^s} \, \frac{1}{(n+|k|)^s}.$$

$\square$

Proof. For $h \in \mathbb{Z}$, let $e_h(x) = \mathrm{e}^{2\pi\, i\, hx}$ for $x \in [0,1]$. Since f is a periodic function from $\widetilde{H}^s$ we can write

$$f(x) = \sum_{h \in \mathbb{Z}} \hat{f}_h \, e_h(x),$$

with the Fourier coefficients $\hat{f}_h = \int_0^1 f(x)\, \mathrm{e}^{-2\pi\, i\, hx}\, \mathrm{d}x$. Since f is smooth the last series is also pointwise convergent. Then

$$A_n^{\mathrm{QMC}}(f) = \frac{1}{n} \sum_{h \in \mathbb{Z}} \hat{f}_h \sum_{j=1}^{n} \left[\mathrm{e}^{2\pi\, i\, (h-k)/n} \right]^j.$$

Note that the sum with respect to j is zero if $h - k \neq 0 \mod n$, and is equal to n if $h - k = 0 \mod n$. Therefore we can restrict h to $h = k + jn$ with $j \in \mathbb{Z}$, and

$$A_n^{\mathrm{QMC}}(f) = \sum_{j \in \mathbb{Z}} \hat{f}_{k+jn}.$$

For $j=0$ we have $\hat{f}_k = I_k(f)$ which yields

$$I_k(f) - A_n^{\rm QMC}(f) = - \sum_{j\in\mathbb{Z}\setminus 0} \hat{f}_{k+jn}.$$

Let $a_h = \max\{1, (2\pi h)^{2s}\}$. Clearly $a_h = a_{-h}$. Since e_h's are orthogonal in $\widetilde{H}^s$ and $\|e_h\|_{H^s}^2 = a_h$, for $f \in \widetilde{H}^s$ we have

$$\|f\|_{H^s}^2 = \sum_{h\in\mathbb{Z}} |\hat{f}_h|^2 a_h < \infty.$$

Hence,

$$|I_k(f) - A_n^{\rm QMC}(f)| = \left| \sum_{j\in\mathbb{Z}\setminus 0} \hat{f}_{k+jn}\, a_{k+jn}^{1/2}\, a_{k+jn}^{-1/2} \right| \le \left(\sum_{j\in\mathbb{Z}\setminus 0} |\hat{f}_{k+jn}|^2 a_{k+jn} \right)^{1/2} \left(\sum_{j\in\mathbb{Z}\setminus 0} a_{k+jn}^{-1} \right)^{1/2}$$

and

$$|I_k(f) - A_n^{\rm QMC}(f)| \le \|f\|_{H^s} \left(\sum_{j=1}^{\infty} \left(\frac{1}{\max\{1, (2\pi(jn+k))^{2s}\}} + \frac{1}{\max\{1, (2\pi(jn-k))^{2s}\}} \right) \right)^{1/2}.$$

The last inequality becomes an equality if we take

$$f = c \sum_{j\in\mathbb{Z}\setminus 0} \hat{f}_{k+jn} e_{k+jn} \quad \text{with} \quad c \ne 0, \quad \text{and} \quad \hat{f}_{k+jn} = a_{k+jn}^{-1}.$$

We can choose c such that $\|f\|_{H^s} = 1$. This yields the formula for $\tilde{e}(A_n^{\rm QMC})$ and proves (i).

Using (i), we obtain for $n \in [1, |k|]$ that

$$\tilde{e}(A_n^{\rm QMC}) > \left(\frac{1}{\max\{1, (2\pi(|k|-n))^{2s}\}} \right)^{1/2} > \left(\frac{1}{\max\{1, (2\pi k)^{2s}\}} \right)^{1/2} = \tilde{e}(0,k,s).$$

This proves (ii).

We now estimate $\tilde{e}(A_n^{\rm QMC})$ for $n > |k|$. For such an integer n and all $j \in \mathbb{N}$ we have

$$\frac{1}{\max\{1, (2\pi(jn+k))^{2s}\}} + \frac{1}{\max\{1, (2\pi(jn-k))^{2s}\}} \le \frac{2}{(2\pi(jn-|k|))^{2s}}.$$

This yields

$$\tilde{e}^2(A_n^{\rm QMC}) \;\le\; \frac{2}{(2\pi)^{2s}} \sum_{j=1}^{\infty} \frac{1}{(jn-|k|)^{2s}}.$$

We have

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{1}{(jn-|k|)^{2s}} &= \frac{1}{(n-|k|)^{2s}} + \sum_{j=2}^{\infty} \frac{1}{(jn-|k|)^{2s}} \\
&\le \frac{1}{(n-|k|)^{2s}} + \int_1^{\infty} \frac{\mathrm{d}x}{(nx-|k|)^{2s}} \\
&= \frac{1}{(n-|k|)^{2s}} - \frac{1}{(2s-1)\,n}\,(nx-|k|)^{-(2s-1)}\Big|_1^{\infty} \\
&= \frac{1}{(n-|k|)^{2s}} + \frac{1}{(2s-1)\,n\,(n-|k|)^{2s-1}} \\
&= \left(1+\frac{n-|k|}{(2s-1)\,n}\right)\frac{1}{(n-|k|)^{2s}} \le \frac{2s}{2s-1}\,\frac{1}{(n-|k|)^{2s}} \\
&\le \frac{2}{(n-|k|)^{2s}}.
\end{aligned}$$

This completes the estimate of $\tilde{e}(A_n^{\rm QMC})$ for $n > |k|$, and proves (iii).

If $n > [(1+\alpha)/(1-\alpha)]\,|k|$ we have $n-|k| \ge \alpha(n+|k|)$ and $(n-|k|)^{-s} \le \alpha^{-s}(n+|k|)^{-s}$. Then (iii) easily yields (iv) and completes the proof. □

We comment on Theorem 4. Note that for $k = 0$, the point (ii) cannot happen and the assumptions of (iii) and (iv) always hold. We now discuss this theorem for $k \ne 0$. We start with (iv). Obviously, if α is close to zero then the condition on n is relaxed. However, the upper bound on $\tilde{e}(A_n^{\rm QMC})$ is weaker since the factor $(2\pi\alpha)^{-s}$ goes to infinity. On the other hand, if α goes to one than the condition on n is more severe but the upper bound (in terms of $(n+|k|)^{-s}$) on $\tilde{e}(A_n^{\rm QMC})$ is better. This means that there is a tradeoff between the condition on n and the quality of the upper bound on $\tilde{e}(A_n^{\rm QMC})$. This problem disappears if n goes to infinity. Then we can take α close to one. In fact, the formula for $\tilde{e}(A_n^{\rm QMC})$ for n tending to infinity yields

$$\lim_{n\to\infty} \tilde{e}(A_n^{\rm QMC})\, n^s = \frac{(2\zeta(2s))^{1/2}}{(2\pi)^s}, \tag{13}$$

where ζ is the zeta Riemann function, $\zeta(x) = \sum_{j=1}^{\infty} j^{-x}$ for $x > 1$.

Remark 5. It is interesting that the right hand side of (13) appears for other problems. First of all, it is the norm of the embedding of $\widetilde{H}^s \cap \{f \in \widetilde{H}^s | \int_0^1 f(x)\,\mathrm{d}x) = 0\}$, equipped with the norm $\|f^{(s)}\|_{L_2}$, into $C([0,1])$, see [7, Theorem 1.2]. This was proven by calculating the diagonal values of the corresponding reproducing kernel. Moreover, the right hand side of (13) equals $\frac{1}{s!}\|B_s\|_{L_2}$, where B_s is the Bernoulli polynomial of degree s, see [7, Lemma 2.16]. $\square$

We now comment on Theorem 4 when $n \in [1, |k|]$. In this case we know that the algorithm A_n^{QMC} is even worse than the zero algorithm. For instance, take $n = |k|$. Then it is easy to conclude from (i) that

$$\tilde{e}(A_n^{\mathrm{QMC}})^2 > 1 + \frac{1}{(4\pi k)^{2s}}. \tag{14}$$

Note that (14) is almost worst possible since every quadrature rule A_n with positive weights which sum up to one satisfies

$$\begin{aligned}\tilde{e}(A_n) \;&\leq\; \sup_{f\in\widetilde{H}^s,\, \|f\|_{H^s}\leq 1} (|I_k(f)| + |A_n(f)|) \leq \|I_k\|_{\widetilde{H}^s\to\mathbb{C}} + \sup_{f\in\widetilde{H}^s,\, \|f\|_{H^s}\leq 1}\ \sup_{x\in[0,1]} |f(x)| \\ &=\; \frac{1}{(2\pi|k|)^s} + \|\mathrm{Id}\|.\end{aligned}$$

Here, $\mathrm{Id} : \widetilde{H}^s \to C([0,1])$ is the embedding operator, i.e., $\mathrm{Id}f = f$ for all $f \in H^s$. We can estimate its norm as follows. We know that $\widetilde{H}^s$ is a reproducing kernel Hilbert space with the kernel[1]

$$\widetilde{K}_s(x,t) \;=\; 1 + (-1)^{s-1}\, B_{2s}^*(\{x-t\}) \;=\; 1 + \sum_{h\in\mathbb{Z}\setminus 0} \frac{\mathrm{e}^{2\pi\, i\, h(x-t)}}{(2\pi h)^{2s}}$$

where $B_k^* = B_k/k!$ is the kth normalized Bernoulli polynomial, see (22), and $\{x-t\}$ is the fractional part of $x-t$.

This implies for f with $\|f\|_{H^s} \leq 1$ that

$$f^2(x) = \left\langle f, \widetilde{K}_s(\cdot,x)\right\rangle_s^2 \leq \|f\|_{H^s}^2\, \widetilde{K}_s(x,x) \leq 1 + \frac{2}{(2\pi)^{2s}} \sum_{j=1}^{\infty} \frac{1}{j^{2s}}.$$

Hence, $\|\mathrm{Id}\|^2 \leq 1 + 2\zeta(2s)/(2\pi)^{2s}$ and

$$\tilde{e}(A_n) \leq 1 + \frac{1}{(2\pi|k|)^s} + \frac{(2\zeta(2s))^{1/2}}{(2\pi)^s}$$

[1] The formula of the reproducing kernel of H^s given as (10.2.4) on page 130 in [14] and as Example 21 on page 320 in [2] has a typo. The term $B_s^*(x)B_s^*(t)$ should be replaced by $\sum_{j=1}^s B_j^*(x)B_j^*(t)$, as is correctly stated in the original paper [3], where this result is proved.

which for large s is close to one as the right hand-side of (14).

We now show how to modify the algorithm $A_n^{\rm QMC}$ such that its worst case error is smaller than the initial error $\tilde{e}(0,k,s)$ with no condition on n. It turns out that the weight n^{-1} used by the algorithm $A_n^{\rm QMC}$ is too large.

Theorem 6. For $a \in \mathbb{R}$, consider the algorithm of the form

$$A_{n,a}(f) = \frac{a}{n} \sum_{j=1}^{n} f(j/n) \, \exp^{-2\pi\, i\, k\, j/n} \qquad \text{for all} \qquad f \in \tilde{h}^s.$$

The worst case error of $A_{n,a}$ is minimized with respect to a for

$$a = a_n^* = \frac{[\tilde{e}(0,k,s)]^2}{[\tilde{e}(0,k,s)]^2 + [\tilde{e}(A_n^{\rm QMC})]^2},$$

and

$$\tilde{e}(A_{n,a_n^*}) = \frac{\tilde{e}(0,k,s)\ \tilde{e}(A_n^{\rm QMC})}{\sqrt{[\tilde{e}(0,k,s)]^2 + [\tilde{e}(A_n^{\rm QMC})]^2}}.$$

Clearly,

$$a_n^* < 1 \qquad \text{and} \qquad \tilde{e}(A_{n,a_n^*}) < \min\{\tilde{e}(0,k,s), \tilde{e}(A_n^{\rm QMC})\}.$$

□

Proof. Repeating the analysis of the first part of the proof of Theorem 4, we obtain

$$I_k(f) - A_{n,a}(f) = (1-a)\hat{f}_k - a \sum_{j \in \mathbb{Z}\setminus 0} \hat{f}_{k+jn}.$$

Similarly as before we use $a_h = \max\{1, (2\pi h)^{2s}\}$ and conclude that

$$\tilde{e}(A_{n,a}) = \left(\frac{(1-a)^2}{a_k} + a^2 \sum_{j \in \mathbb{Z}\setminus 0} \frac{1}{a_{k+jn}} \right)^{1/2} = \left((1-a)^2 [\tilde{e}(0,k,s)]^2 + a^2 [\tilde{e}(A_n^{\rm QMC})]^2 \right)^{1/2}.$$

Clearly, the last expression is minimized with respect to a for $a = a_n^*$ from which we obtain the form of $\tilde{e}(A_{n,a_n^*})$. This completes the proof. □

We discuss a_n^* which decreases the weight n^{-1} in the algorithm A_{n,a_n^*}. For $n \in [1,|k|]$, the point (ii) of Theorem 4 yields that $a_n^* < 1/2$. For $n = |k| \geq 1$ we know from (14) that $\tilde{e}(A_n^{\mathrm{QMC}}) > 1$, and therefore $a_n^* \leq [\tilde{e}(0,k,s)]^2 = (2\pi|k|)^{-2s}$ which is polynomially small in $|k|$ and exponentially small in s. On the other hand, if k is fixed and n goes to infinity then a_n^* goes to one and the algorithm A_{n,a_n^*} becomes the same as the algorithm A_n^{QMC}.

The algorithm A_{n,a_n^*} has a (small) computational drawback since it requires the exact value of a_n^* which is given by the infinite series describing the worst case error of $\tilde{e}(A_n^{\mathrm{QMC}})$. Of course, it can be precomputed to an arbitrary accuracy.

There is another simple idea how to modify the algorithm A_n^{QMC} without computing a_n^*. Namely, for small n we use the zero algorithm whereas for large n we use the algorithm A_n^{QMC}. More precisely, for $n = 0, 1, \dots,$ we define the algorithm

$$A_n^*(f) = \begin{cases} 0 & \text{if } \ n = 0 \ \text{ or } \ n < 2|k|, \\ A_n^{\mathrm{QMC}}(f) & \text{if } \ n \geq \max(1, 2|k|). \end{cases} \tag{15}$$

The algorithm A_n^* uses no information on f if $n = 0$ or $n < 2|k|$, and n function values otherwise. Based on Theorem 4 and the discussion after its proof it is easy to show

Corollary 7. We have

$$\tilde{e}(A_n^*) \begin{cases} = 1 & \text{for} \quad k = 0 \ \text{ and } \ n = 0, \\ = \frac{1}{(2\pi|k|)^s} & \text{for} \quad k \neq 0 \ \text{ and } \ n \in [0, 2|k|), \\ \leq \frac{2}{(2\pi)^s} \frac{1}{(n-|k|)^s} & \text{for} \quad n \geq \max(1, 2|k|). \end{cases}$$

Furthermore,

$$\tilde{e}(A_n^*) \ \leq \ \left(\frac{3}{2\pi}\right)^s \frac{2}{(n+|k|)^s} \qquad \text{for all} \qquad n \geq 1. \tag{16}$$

□

Proof. Assume first that $k = 0$. Then for $n = 0$ we have $A_0^* = 0$ and $\tilde{e}(A_0^*) = 1$. For $n \geq 1$ we have $A_n^* = A_n^{\mathrm{QMC}}$ and we use Theorem 4(iii) to get the third estimate on $\tilde{e}(A_n^*)$.

Assume now that $k \neq 0$. For $n \in [0, 2|k|)$ the error of $A_0^* = 0$ is the initial error which is $(2\pi|k|)^{-s}$. For $n \geq 2|k|$ we have $A_n^* = A_n^{\mathrm{QMC}}$ and the estimate on $\tilde{e}(A_n^*)$ follows from Theorem 4(iii).

We now prove the estimate (16). Again assume first that $k = 0$. Consider first the case $n \geq \max(1, 2|k|)$. We can now apply Theorem 4(iv) with $\alpha = 1/3$ and then

$$\tilde{e}(A_n^*) = \tilde{e}(A_n^{\mathrm{QMC}}) \leq \left(\frac{3}{2\pi}\right)^s \frac{2}{(n+|k|)^s},$$

as claimed. It remains to consider the case $n \in [1, 2|k|)$ for $k \neq 0$. Then $|k| > (n+|k|)/3$ and

$$\tilde{e}(A_n^*) = \frac{1}{(2\pi|k|)^s} \le \left(\frac{3}{2\pi(n+|k|)}\right)^s = \left(\frac{3}{2\pi}\right)^s \frac{1}{(n+|k|)^s},$$

as claimed. This completes the proof. $\square$

We stress that all algorithms considered so far use only function values although we allow also computation of derivatives up to order $s-1$. Furthermore, they use function values at equally spaced points and use the same weights n^{-1} or $a_n^* n^{-1}$ for large n. Although algorithms that minimize the worst case error are probably not of this form, we now prove a lower bound on the order of convergence of an arbitrary algorithm, and show that this order is $(n+|k|)^s$. Hence the algorithm A_n^* enjoys the best possible order of convergence. Additionally, the algorithm A_n^* is easy to implement.

Theorem 8. Consider the integration problem I_k defined over the space $\widetilde{H}^s$ of periodic functions with $s \in \mathbb{N}$. Let $\tilde{e}(n,k,s)$ be the nth minimal worst case error of all algorithms that use at most n function or derivatives (up to order $s-1$) values, see (10). There is a number $c_s > 0$ such that

$$\frac{c_s}{(n+|k|)^s} \;\le\; \tilde{e}(n,k,s) \;\le\; \left(\frac{3}{2\pi}\right)^s \frac{2}{(n+|k|)^s}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. $\square$

Proof. The upper bound has been already shown for the algorithm A_n^*. Hence, we only need to prove the lower bound.

Let A_n be an arbitrary algorithm of the form (8) that uses $f^{(\ell_j)}(x_j)$ for some $\ell_j \in [0, s-1]$ and $x_j \in [0,1]$ for $j = 1, 2, \ldots, n$. Suppose that for $f \in \widetilde{H}^s$ we get $f^{(\ell_j)}(x_j) = 0$ for all $j = 1, 2, \ldots, n$. Since $-f$ also belongs to $\widetilde{H}^s$, the algorithm A_n cannot distinguish between $I_k(f)$ and $I_k(-f) = -I_k(f)$. Therefore $|I_k(f)|$ is a lower bound on the worst case error of A_n. This leads to a well-known inequality

$$\tilde{e}(A_n) \ge \sup\{|I_k(f)| \;:\; f \in \widetilde{H}^s,\, \|f\|_{H^s} \le 1,\, N(f) = 0\},$$

where

$$N(f) = [f^{(\ell_j)}(x_j),\; j = 1, 2, \ldots, n]$$

Below we will construct a function f with large $|I_k(f)|$ and all of the $s \cdot n$ values $f^{(\ell)}(x_j), j = 1, \ldots, n,\ \ell = 0, \ldots, s-1$, are equal to zero. Obviously, such a function f satisfies $N(f) = 0$.

We consider a real-valued f and the real part

$$\widehat{I}_k(f) = \int_0^1 f(x)\cos(2\pi kx)\,\mathrm{d}x$$

of $I_k(f)$. Define the disjoint subintervals $T_{i,k} \subset [0,1]$ for which $|\cos(2\pi kx)| \ge 1/\sqrt{2}$. There are $2|k|+1$ such intervals. For $k=0$ we have $T_{1,0} = [0,1]$, whereas for $k \neq 0$ the lengths of $T_{i,k}$'s for $i = 1,2,\dots,2|k|+1$ are $\frac{1}{8|k|}, \frac{1}{4|k|}, \dots, \frac{1}{4|k|}, \frac{1}{8|k|}$ with the total length $1/2$. The points $x_1,\dots,x_n$ used by A_n may divide the $T_{i,k}$ further and altogether we obtain $m \in [2|k|+1, 2|k|+1+n]$ intervals $\widehat{T}_{1,k}\dots,\widehat{T}_{m,k}$; all the endpoints of the $\widehat{T}_{i,k}$ coincide with an endpoint of one of the $T_{i,k}$ or are one of the x_j. Again, the sum of the lengths of the $\widehat{T}_{i,k}$ is $1/2$.

We define $\Phi(x) = d_s(\cos^2(\pi x/2))^s$ for $|x| \le 1$ and $\Phi(x) = 0$ otherwise. Then $\Phi \in C^s(\mathbb{R})$ and we can choose $d_s > 0$ in such a way that $\|\Phi\|_{H^s([-1,1])} = 1$.

Let the length of the interval $\widehat{T}_{i,k}$ be $1/n_i$ and let y_i be its midpoint. For $i = 1,2,\dots,m$, we define a scaled version of Φ by

$$\Phi_i(x) = \frac{\mathrm{sgn}(\cos(2\pi k y_i))}{(2n_i)^s}\,\Phi(2n_i x - 2n_i y_i) \qquad \text{for all} \qquad x \in \mathbb{R}.$$

Note that the support of Φ_i is $\widehat{T}_{i,k}$ and $\|\Phi_i\|_{H^s([-1,1])} \le 1$. Furthermore,

$$\begin{aligned}\widehat{I}_k(\Phi_i) &= \frac{1}{(2n_i)^s}\int_{\widehat{T}_{i,k}} |\cos(2\pi kx)|\Phi(2n_i(x-y_i))\,\mathrm{d}x \ge \frac{1}{2^{s+1/2}\,n_i^s}\int_{\widehat{T}_{i,k}} \Phi(2n_i(x-y_i))\,\mathrm{d}x \\ &= \frac{d_s}{2^{s+3/2}\,n_i^{s+1}}\int_{-1}^1 (\cos^2(\pi t/2))^s\,\mathrm{d}t.\end{aligned}$$

Finally we define our "fooling function" by

$$f = \sum_{i=1}^m \Phi_i.$$

It is easy to check that $f \in \widetilde{H}^s$ with $N(f) = 0$ and $\|f\|_{H^s} \le 1$. We can also estimate the integral and obtain

$$|I_k(f)| \ge |\widehat{I}_k(f)| = \sum_{i=1}^m \widehat{I}_k(\Phi_i) \ge \tilde{c}_s \sum_{i=1}^m n_i^{-s-1}$$

with

$$\tilde{c}_s = \frac{d_s}{2^{s+3/2}}\int_{-1}^1 (\cos^2(\pi t/2))^s\,\mathrm{d}t > 0.$$

It is easy to check by standard means that

$$\min_{n_i\colon \sum_{i=1}^m n_i^{-1}=1/2} \sum_{i=1}^m n_i^{-s-1} = \frac{1}{2^{s+1}}\,\frac{1}{m^s} \geq \frac{1}{2^{s+1}\,(2|k|+1+n)^s} \geq \frac{1}{2\cdot 4^s (n+|k|)^s}.$$

This proves the lower bound with $c_s = \tilde{c}_s/(2\cdot 4^s)$. $\square$

We stress that the lower bound in Theorem 8 holds for a larger class of algorithms than the class (8) for $s>1$. Namely it holds for algorithms

$$A_n(f) = \sum_{j=1}^n \sum_{\ell=1}^{s-1} a_{j,\ell}\, f^{(\ell)}(x_j)$$

for arbitrary $a_{j,\ell} \in \mathbb{C}$ and $x_j \in [0,1]$. That is, we now use $n \cdot s$ values of f and its derivatives instead of n, however, we still have "only" n sample points to choose.

Theorem 8 states that both lower and upper bounds on the nth minimal error decay with $|k|$. Does it really mean that high oscillation makes the problem easy? The answer to this question depends on whether we consider the absolute or normalized error criterion.

For the *absolute error criterion*, the information complexity $\tilde{n}^{\rm abs}(\varepsilon,k,s)$ is defined as the minimal n for which the error is at most $\varepsilon\in(0,1)$. That is,

$$\tilde{n}^{\rm abs}(\varepsilon,k,s) = \min\{\, n \mid \ \tilde{e}(n,k,s) \leq \varepsilon\,\}.$$

Clearly, $\tilde{n}^{\rm abs}(\varepsilon,k,s)=0$ for $\varepsilon \geq \tilde{e}(0,k,s)$ since we can solve the problem by the zero algorithm. For $\varepsilon < \tilde{e}(0,k,s)$ we can bound $\tilde{n}^{\rm abs}(\varepsilon,k,s)$ by Theorem 8. This implies the following corollary.

Corollary 9. Consider the absolute error criterion for the integration problem I_k defined over the periodic space $\widetilde{H}^s$. Let c_s be from Theorem 8.

- For $k=0$ and all $\varepsilon\in(0,1)$ we have

$$c_s^{1/s}\left(\frac{1}{\varepsilon}\right)^{1/s} \leq \tilde{n}^{\rm abs}(\varepsilon,0,s) \leq \left\lceil \frac{3}{2\pi}\left(\frac{\sqrt{2}}{\varepsilon}\right)^{1/s}\right\rceil.$$

- For $k\neq 0$ and $\varepsilon \in [1/(2\pi|k|)^s, 1)$ we have

$$\tilde{n}^{\rm abs}(\varepsilon,k,s) = 0,$$

whereas for $\varepsilon \in (0, 1/(2\pi|k|)^s)$ we have

$$c_s^{1/s} \left(\frac{1}{\varepsilon}\right)^{1/s} - |k| \le \tilde{n}^{\mathrm{abs}}(\varepsilon, k, s) \le \left\lceil \frac{3}{2\pi} \left(\frac{\sqrt{2}}{\varepsilon}\right)^{1/s} \right\rceil - |k|.$$

$\square$

This means that for the absolute error criterion the problem becomes easier for large $|k|$, but the asymptotic behavior of $\tilde{n}^{\mathrm{abs}}(\varepsilon, k, s)$, as $\varepsilon \to 0$, does not depend on k.

We now turn to the *normalized error criterion* in which we want to reduce the initial error $\tilde{e}(0, k, s)$ by a factor $\varepsilon \in (0, 1)$. That is, the information complexity $\tilde{n}^{\mathrm{nor}}(\varepsilon, k, s)$ is defined as

$$\tilde{n}^{\mathrm{nor}}(\varepsilon, k, s) = \min \{ n \mid \tilde{e}(n, k, s) \le \varepsilon \, \tilde{e}(0, k, s) \} .$$

In this case we always have $\tilde{n}^{\mathrm{nor}}(\varepsilon, k, s) \ge 1$. Note that for $k = 0$ we have $\tilde{e}(0, 0, s) = 1$ and there is no difference between the normalized and absolute error criteria.

For $k \ne 0$ the situation is quite different. From Theorem 4, Theorem 8 and Proposition 3 it is easy to prove the following corollary.

Corollary 10. Consider the normalized error criterion for the integration problem I_k defined over the periodic space $\widetilde{H}^s$. Let c_s be from Theorem 8.

For all $k \ne 0$ and all $\varepsilon \in (0, 1)$ we have

$$|k| \left(2\pi \left(\frac{c_s}{\varepsilon}\right)^{1/s} - 1\right) \le \tilde{n}^{\mathrm{nor}}(\varepsilon, k, s) \le |k| \left\lceil 3 \left(\frac{\sqrt{2}}{\varepsilon}\right)^{1/s} - 1 \right\rceil ,$$

which can be written as

$$\tilde{n}^{\mathrm{nor}}(\varepsilon, k, s) = \Theta\left(\frac{|k|}{\varepsilon^{1/s}}\right) \qquad \text{as} \qquad \varepsilon \to 0.$$

$\square$

Hence, for the normalized error criterion the problem becomes harder for large $|k|$. It is interesting that the dependence on $|k|$ is linear and does not depend on s. In particular, for fixed s and fixed $\varepsilon < (2\pi)^s c_s$ we have

$$\lim_{|k| \to \infty} \tilde{n}^{\mathrm{nor}}(\varepsilon, k, s) = \infty. \tag{17}$$

4 The non-periodic case

We now turn to the case of non-periodic functions, i.e., we consider the Sobolev space

$$H^s = \{f : [0,1] \to \mathbb{C} \mid f^{(s-1)} \text{ is abs. cont., } f^{(s)} \in L_2\} \tag{18}$$

for a finite $s \in \mathbb{N}$. The inner product $\langle \cdot, \cdot \rangle_s$ in H^s is again defined by (2).

Clearly, for all $j = 1, 2, \ldots, s$ we have $H^s \subset H^j$ and $\|f\|_{H^j} \le \|f\|_{H^s}$ for all $f \in H^s$. This follows from the inequality

$$\int_0^1 |f'(x)|^2 \,\mathrm{d}x \;\ge\; \int_0^1 |f(x)|^2 \,\mathrm{d}x \;-\; \left(\int_0^1 f(x) \,\mathrm{d}x\right)^2$$

for differentiable functions f and implies that the unit ball of H^s is a subset of the unit ball of H^j.

Again we want to approximate the integral

$$I_k(f) := \int_0^1 f(x)\, \mathrm{e}^{-2\pi i\, kx} \,\mathrm{d}x, \tag{19}$$

where $k \in \mathbb{Z}$ and $f \in H^s$ with $s \in \mathbb{N}$. Without loss of generality we consider linear algorithms A_n of the form (8). Similarly as before, we define the worst case error of A_n as

$$e(A_n) \;:=\; \sup_{f \in H^s,\, \|f\|_{H^s} \le 1} |I_k(f) - A_n(f)|,$$

and the nth minimal worst case error as

$$e(n, k, s) \;:=\; \inf_{A_n} e(A_n).$$

In particular, the initial error is given by

$$e(0, k, s) := \sup_{f \in H^s, \|f\|_{H^s} \le 1} |I_k(f)| = \|I_k\|_{H^s \to \mathbb{C}},$$

compare with (10) and (11). We do not now use the tilde to stress the non-periodic case.

Note that H^s is obviously a superset of $\widetilde{H}^s$ and hence, lower bounds that were proved in Section 3 for $\widetilde{H}^s$ also hold for H^s, i.e.,

$$e(n, k, s) \ge \tilde{e}(n, k, s). \tag{20}$$

We start with the computation of the initial error. As we shall see, for large s and $|k|$, it is now much larger than for the periodic case. In particular, the initial error for $k \neq 0$ does not tend to zero if s tends to infinity.

Similarly to (12) we want to compute the representer $h_{k,s}$ of I_k in H^s. Using the same functions $e_k(x) = \mathrm{e}^{2\pi i k x}$, which satisfy $\|e_k\|_{H^s} = (2\pi|k|)^s$, we obtain

$$\begin{aligned}\langle f, e_k \rangle_s &= \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = (-1)^s \left\langle f, e_k^{(2s)} \right\rangle_0 + \sum_{\ell=0}^{s-1} (-1)^\ell \left[f^{(s-\ell-1)} \overline{e_k^{(s+\ell)}} \right]_0^1 \\ &= (2\pi k)^{2s}\, I_k(f) + (-1)^s \sum_{\ell=1}^{s} (2\pi i k)^{2s-\ell} \left(f^{(\ell-1)}(1) - f^{(\ell-1)}(0) \right).\end{aligned} \tag{21}$$

Surprisingly, the functionals $f^{(\ell-1)}(1) - f^{(\ell-1)}(0)$, $\ell = 1, \dots, s$, or more precisely their representers in H^s, have some nice properties that will be useful in the following analysis. These representers are given by the normalized Bernoulli polynomials

$$B_\ell^*(x) \;=\; \frac{1}{\ell!}\, B_\ell(x), \tag{22}$$

where the Bernoulli polynomials B_ℓ, $\ell \geq 0$, are the unique polynomials that are given by

$$\int_t^{t+1} B_\ell(x)\,\mathrm{d}x \;=\; t^\ell \quad \text{for all} \quad t \in \mathbb{R} \;\; \text{with} \;\; 0^0 = 1.$$

To see this note that $[B_\ell^*]' = B_{\ell-1}^*$ as well as $\int_0^1 B_\ell^*(x)\,\mathrm{d}x = 0$, $\ell \geq 1$, and $\int_0^1 B_0^*(x)\,\mathrm{d}x = 1$. In particular, this implies for $f \in H^s$ and $\ell \leq s$ that

$$\langle f, B_\ell^* \rangle_s = \left\langle f^{(\ell)}, 1 \right\rangle_0 = f^{(\ell-1)}(1) - f^{(\ell-1)}(0),$$

which proves the claim. Additionally, this shows

$$\|B_\ell^*\|_{H^s} = 1 \qquad \text{and} \qquad \langle B_\ell^*, B_m^* \rangle_s = 0 \tag{23}$$

for $\ell, m \in \{0, 1, \dots, s\}$ with $\ell \neq m$ and, consequently,

$$H^s \;=\; \widetilde{H}^s \oplus \{B_1^*\} \oplus \cdots \oplus \{B_s^*\},$$

see e.g. [14, Section 10.2].

Using (21) we obtain the following proposition.

Proposition 11. Let $k \neq 0$. The representer of I_k in H^s is

$$h_{k,s}(x) = (2\pi k)^{-2s}\, \mathrm{e}^{2\pi\, i\, kx} \;-\; \sum_{\ell=1}^{s} (-1)^{\ell} (2\pi i k)^{-\ell}\, B_{\ell}^{*}(x)$$

and the initial error is

$$e(0,k,s) = \|h_{k,s}\|_{H^s} = \sqrt{\frac{2}{\big(2\pi k\big)^{2s}} + \sum_{\ell=1}^{s-1} \frac{1}{\big(2\pi k\big)^{2\ell}}} \;=\; \frac{\beta_{k,s}}{2\pi|k|}$$

with $\beta_{k,1} = \sqrt{2}$ and

$$1 \;\le\; \beta_{k,s} \;=\; \sqrt{\frac{(2\pi k)^{2s} + (2\pi k)^2 - 2}{(2\pi k)^{2s} - (2\pi k)^{2(s-1)}}} \;\le\; \sqrt{1 + \frac{2}{(2\pi k)^2 - 1}} \;\le 1.02566$$

for $s > 1$. Note that $\lim_{k\to\infty} \beta_{k,s} = 1$.

For $k = 0$, the representer is $h_{0,s} = 1$ and the initial error is one, $\tilde{e}(0,0,s) = 1$.

We are ready to discuss algorithms for the non-periodic case. One of the ideas to get such algorithms is first to periodize functions f from H^s by computing $f^{(0)}(0), \dots, f^{(s-1)}(0)$ and $f^{(0)}(1), \dots, f^{(s-1)}(1)$, and then apply the algorithm A^{*}_{n-2s} from Section 3. Of course, this requires to assume that $n \ge 2s$ which is a bad assumption if s is large or even impossible to satisfy if $s = \infty$. Therefore for $n < 2s$ we need to proceed differently. As already discussed $f \in H^s$ implies that $f \in H^j$ for all $j \le s$. Therefore we can use periodization for H^j by computing $f^{(0)}(0), \dots, f^{(j-1)}(0)$ and $f^{(0)}(1), \dots, f^{(j-1)}(1)$ as long as $n \ge 2j$. Then we can again apply the algorithm A^{*}_{n-2j} from Section 3. Formally, this algorithm was studied only for H^s but it is obvious that its error can be also analyzed for H^j with the change of s to j.

Another idea to obtain algorithms for small n relative to s is to use the integration of Taylor's expansion of $f \in H^s$ at $\frac{1}{2}$. As we shall see this approach is appropriate if $|k|$ is relatively small with respect to n. To explain these ideas more precisely we need some preparations.

4.1 Periodization

For $j = 1, 2, \dots, s$ and $f \in H^s$, we compute

$$f^{(0)}(0), \dots, f^{(j-1)}(0) \quad \text{and} \quad f^{(0)}(1), \dots, f^{(j-1)}(1).$$

With this information we define a polynomial $p_{f,j}$ of degree at most j such that $\tilde{f}_j = f - p_{f,j}$ is a periodic function from $\widetilde{H}^j$. To obtain the polynomial $p_{f,j}$, we use the normalized Bernoulli polynomials from (22). In particular, $B_0^*(x) = 1$ and $B_1^*(x) = x - \frac{1}{2}$. For $m \geq 1$, we have $[B_m^*]' = B_{m-1}^*$ which yields

$$[B_m^*]^{(\ell)} = B_{m-\ell}^* \qquad \text{for all} \qquad \ell = 0, 1, \ldots, m. \tag{24}$$

Furthermore,

$$B_1^*(1) - B_1^*(0) = 1 \qquad \text{and} \qquad B_m^*(1) - B_m^*(0) = 0 \quad \text{for all} \quad m \neq 1. \tag{25}$$

For $f \in H^s \subset H^j$, we define the polynomials $p_{f,j}$ by

$$p_{f,j}(x) \ := \ \sum_{m=0}^{j-1} \left(f^{(m)}(1) - f^{(m)}(0) \right) B_{m+1}^*(x). \tag{26}$$

We stress that the computation of the value $p_{f,j}(x)$ requires the $2j$ values of $f^{(m)}(1)$ and $f^{(m)}(0)$ for $m = 0, 1, \ldots, j-1$.

For $\ell = 0, 1, \ldots, j-1$, we conclude from (24) that

$$p_{f,j}^{(\ell)}(x) = \sum_{m=\max(0,\ell-1)}^{j-1} \left(f^{(m)}(1) - f^{(m)}(0) \right) B_{m+1-\ell}^*(x).$$

Using (25) we obtain

$$p_{f,j}^{(\ell)}(1) - p_{f,j}^{(\ell)}(0) \ = \ f^{(\ell)}(1) - f^{(\ell)}(0) \qquad \text{for all} \qquad \ell = 0, 1, \ldots, j-1.$$

This implies that $f - p_{f,j} \in \widetilde{H}^j$ for all $f \in H^s$.

Since $f - p_{f,j} \in \widetilde{H}^j$ and the norm of I_k restricted to the space $\widetilde{H}^j$ is given by Proposition 3 with s replaced by j, we know that

$$|I_k(f) - I_k(p_{f,j})| \ \leq \ \left\| I_k\big|_{\widetilde{H}^j} \right\| \, \|f - p_{f,j}\|_{H^j} \ = \ \frac{\|f - p_{f,j}\|_{H^j}}{\max\{1, (2\pi|k|)^j\}}$$

and, by Parseval's identity (in $\widetilde{H}^j$), that

$$\begin{aligned} \|f - p_{f,j}\|_{H^j} \ &= \ \left| \langle f - p_{f,j}, 1 \rangle_j \right|^2 \ + \ \sum_{\ell \in \mathbb{Z} \setminus 0} \left| \langle f - p_{f,j}, e_\ell \rangle_j \right|^2 (2\pi\ell)^{-2j} \\ &= \ \left| \langle f, 1 \rangle_j \right|^2 \ + \ \sum_{\ell \in \mathbb{Z} \setminus 0} \left| \langle f, e_\ell \rangle_j \right|^2 (2\pi\ell)^{-2j} \ \leq \ \|f\|_{H^j} \ \leq \ \|f\|_{H^s}. \end{aligned} \tag{27}$$

Here we used that $\langle p_{f,j}, e_k \rangle_j = 0$ for all $k \in \mathbb{Z}$. This proves that

$$|I_k(f) - I_k(p_{f,j})| \;\le\; \frac{\|f\|_{H^s}}{\max\{1, (2\pi|k|)^j\}}.$$

Note that the last upper bound is not small for $k = 0$. However, if $k \neq 0$ then for $j \in [1, s]$ for all $f \in H^s$ we have

$$|I_k(f) - I_k(p_{f,j})| \;\le\; \frac{\|f\|_{H^s}}{(2\pi|k|)^j}, \tag{28}$$

which is exponentially small in j.

We now show how to compute $I_k(p_{f,j})$ exactly. Indeed,

$$I_k(p_{f,j}) = \sum_{m=0}^{j-1} \big(f^{(m)}(1) - f^{(m)}(0)\big)\, I_k(B^*_{m+1}),$$

and it is enough to compute $I_k(B^*_{m+1})$. For $k = 0$ we have $I_0(B^*_{m+1}) = \int_0^1 B^*_{m+1}(x)\,\mathrm{d}x = 0$ for all $m = 0, 1, \dots, j-1$. Hence, $I_0(p_{f,j}) = 0$.

For $k \neq 0$, we use the Fourier expansion of the normalized Bernoulli polynomials B^*_{m+1},

$$B^*_{m+1}(x) = -\frac{1}{(2\pi\, i\,)^{m+1}} \sum_{\ell \in \mathbb{Z} \setminus 0} \frac{e^{2\pi\, i\, \ell\, x}}{\ell^{j+1}} \qquad \text{for all} \qquad x \in [0, 1]. \tag{29}$$

This yields

$$I_k(B^*_{m+1}) = -\frac{1}{(2\pi\, i)^{m+1}} \sum_{\ell \in \mathbb{Z} \setminus 0} \frac{1}{\ell^{j+1}} \int_0^1 e^{2\pi\, i\, (\ell - k)\, x}\,\mathrm{d}x = -\frac{1}{(2\pi\, ik)^{m+1}}.$$

Hence,

$$I_k(p_{f,j}) = \begin{cases} 0 & \text{for } \;k = 0 \\ -\sum_{\ell=0}^{j-1} \frac{f^{(\ell)}(1) - f^{(\ell)}(0)}{(2\pi\, ik)^{\ell+1}} & \text{for } \;k \neq 0. \end{cases}$$

For $k \neq 0$, the computation of $I_k(p_{f,j})$ requires the $2j$ values of $f^{(\ell)}(1)$ and $f^{(\ell)}(0)$ for $\ell = 0, 1, \dots, j-1$ which are also needed for the computation of $p_{f,j}(x)$.

4.2 Taylor's Expansion

For $n \in [1, s]$, we use Taylor's expansion of $f \in H^s$ at $\frac{1}{2}$. Let

$$T_{f,n}(x) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(x - \tfrac{1}{2}) + \cdots + \frac{f^{(n-1)}(\frac{1}{2})}{(n-1)!}\,(x - \tfrac{1}{2})^{n-1} \qquad \text{for all} \qquad x \in [0, 1].$$

Then

$$f(x) - T_{f,n}(x) = \frac{(x-\frac{1}{2})^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}\left(\tfrac{1}{2} + t(x-\tfrac{1}{2})\right) \mathrm{d}t \qquad \text{for all} \qquad x \in [0,1].$$

This allows us to estimate $I_k(f - T_{f,n})$ since

$$I_k(f - T_{f,n}) = \frac{1}{(n-1)!} \int_0^1 \int_0^1 \mathrm{e}^{-2\pi\, i\, kx} \, (x-\tfrac{1}{2})^n \, (1-t)^{n-1} f^{(n)}\left(\tfrac{1}{2} + t(x-\tfrac{1}{2})\right) \mathrm{d}t \, \mathrm{d}x,$$

and

$$|I_k(f - T_{f,n})| \le \frac{1}{(n-1)!} \int_0^1 \int_0^1 |x-\tfrac{1}{2}|^n \, |f^{(n)}\left(\tfrac{1}{2} + t(x-\tfrac{1}{2})\right)| \, \mathrm{d}t \, \mathrm{d}x.$$

We now change variables $y = \frac{1}{2} + t(x-\frac{1}{2}) \in [0,1]$ such that $\mathrm{d}y = (x-\frac{1}{2})\mathrm{d}t$ and then

$$|I_k(f) - I_k(T_{f,n})| \le \frac{1}{(n-1)!} \int_0^1 \int_0^1 |x-\tfrac{1}{2}|^{n-1} \, |f^{(n)}(y)| \, \mathrm{d}x \, \mathrm{d}y \le \frac{1}{2^{n-1}\, n!} \, \|f^{(n)}\|_{L_2}.$$

This proves that for $n \in [1,s]$ and all $f \in H^s$ we have

$$|I_k(f) - I_k(T_{f,n})| \le \frac{1}{2^{n-1}\, n!} \, \|f\|_{H^s}. \tag{30}$$

Furthermore, we can compute $I_k(T_{f,n})$ exactly if we know $f(\frac{1}{2}), f'(\frac{1}{2}), \dots, f^{(n-1)}(\frac{1}{2})$. Indeed,

$$I_k(T_{f,n}) = \sum_{\ell=0}^{n-1} f^{(\ell)}(\tfrac{1}{2}) \, \frac{1}{\ell!} \int_0^1 \mathrm{e}^{-2\pi\, i\, kx} \, (x-\tfrac{1}{2})^\ell \, \mathrm{d}x.$$

For $k = 0$, we have

$$I_0(T_{f,n}) = \sum_{\ell=0}^{n-1} f^{(\ell)}(\tfrac{1}{2}) \, \frac{1+(-1)^\ell}{2^{\ell+1}(\ell+1)!}. \tag{31}$$

For $k \ne 0$, we use integration by parts and show that

$$\frac{1}{\ell!} \int_0^1 \mathrm{e}^{-2\pi\, i\, kx} \, (x-\tfrac{1}{2})^\ell \, \mathrm{d}x = \frac{1}{(2\pi\, i\, k)^{\ell+1}} \sum_{m=0}^{\ell} \frac{i^m\,(k\pi)^m((-1)^m-1)}{m!}.$$

Hence for $k \ne 0$, we have

$$I_k(T_{f,n}) = \sum_{\ell=0}^{n-1} \frac{f^{(\ell)}(\frac{1}{2})}{(2\pi\, i\, k)^{\ell+1}} \sum_{m=0}^{\ell} \frac{i^m\,(k\pi)^m((-1)^m-1)}{m!}. \tag{32}$$

4.3 Algorithms

With the preparations done in the previous two subsections, we are ready to define algorithms for the non-periodic case.

- Assume first that $k \in \mathbb{Z} \setminus \{0\}$.

We discuss algorithms based on periodization for $f \in H^s$. We define the algorithm A_n^{Per} for all even $n \in [2, 2s)$ and for $n = 2s + \ell$ with $\ell \in \mathbb{N}_0$.

For even $n \in [2, 2s)$ we compute $f^{(0)}(0), \dots, f^{((n-2)/2)}(0)$, $f^{(0)}(1), \dots, f^{((n-2)/2)}(1)$, and define

$$A_n^{\mathrm{Per}}(f) = I_k(p_{f,n/2})$$

with $p_{f,n/2}$ given by (26) for $j = n/2 \le s$.

For $n = 2s + \ell$ with $\ell \in \mathbb{N}_0$, we compute $f^{(0)}(0), \dots, f^{(s-1)}(0)$, $f^{(0)}(1), \dots, f^{(s-1)}(1)$ to obtain the polynomial $p_{f,s}$. Then we define

$$A_n^{\mathrm{Per}}(f) = I_k(p_{f,s}) + A_{\ell+1}^*(f - p_{f,n})$$

with the algorithm $A_{\ell+1}^*$ from Section 3 defined by (15). The algorithm $A_{\ell+1}^*$ uses no extra information on f if $\ell < 2|k| - 1$. For $\ell \ge 2|k| - 1$, the algorithm $A_{\ell+1}^*$ uses extra ℓ function values at $j/(\ell+1)$ for $j = 1, 2, \dots, \ell$. Note that we have already computed the function value at $j/(\ell+1)$ for $j = \ell + 1$.

We stress that the algorithm A_n^{Per} is well defined for $n = 2s + \ell$ since $f - p_f \in \widetilde{H}^s$ and $\widetilde{H}^s$ is the domain of the algorithm $A_{\ell+1}^*$. For $f \in \widetilde{H}^s$ we have $p_{f,j} = 0$ for all $j \in [1, s]$, and therefore $A_n^{\mathrm{Per}}(f) = 0$ for all even $n \in [2, 2s)$ and $A_n^{\mathrm{Per}}(f) = A_{\ell+1}^*(f)$ for $n = 2s + \ell$.

The algorithm A_n^{Per} uses at most n evaluations of f. Indeed, for even $n \in [2, 2s)$ it uses $n/2$ evaluations at the endpoint points $x = 0$ and $x = 1$, so that the total number is n. For $n = 2s + \ell$, the algorithm A_n^{Per} uses two function values and $2(s-1)$ values of derivatives of f at $x = 0$ and $x = 1$, as well as at most ℓ functions values at $j/(\ell+1)$ for $j = 1, \dots, \ell$, which is $2 + 2(s-1) + \ell = 2s + \ell = n$, as claimed.

From the formulas of Sections 3 and 4.1 we find the explicit form of A_n^{Per}. For even $n \in [2, 2s)$ and $n = 2s + \ell$ with $\ell < 2|k| - 1$ we have

$$A_n^{\mathrm{Per}}(f) = \sum_{j=0}^{(n-2)/2} \frac{f^{(j)}(0) - f^{(j)}(1)}{(2\pi i\, k)^{j+1}},$$

whereas for $n = 2s + \ell$ with $\ell \geq 2|k| - 1$, we have

$$
\begin{aligned}
A_n^{\mathrm{Per}}(f) \;\; = \;\; & \sum_{j=0}^{s-1} \frac{f^{(j)}(0) - f^{(j)}(1)}{(2\pi i\, k)^{j+1}} \; + \\
& \frac{1}{\ell+1} \sum_{j=1}^{\ell+1} \left(f\left(\frac{j}{\ell+1}\right) - p_{f,s}\left(\frac{j}{\ell+1}\right)\right) \exp^{-2\pi\, i\, k\, j/(\ell+1)}\,.
\end{aligned}
$$

Note that for $s = 1$ the algorithm A_n^{Per} uses only function values since $p_{f,1}(x) = (f(1) - f(0))(x - \frac{1}{2})$, whereas for $s \geq 2$ it also uses derivatives of f. The weights used by the algorithm A_n^{Per} are complex. However, the sum of their absolute values is bounded by an absolute constant independent of n since it is known that the values of the normalized Bernoulli polynomials B_j^*, which are present in $p_{f,s}$, are exponentially small in j. This implies numerical stability of the algorithm A_n.

Obviously, the derivatives $f^{(j)}(0)$ and $f^{(j)}(1)$ for $j = 1, 2, \ldots, s-1$ can be approximated with an arbitrary precision by computing $2(s-1)$ extra function values. Hence, there are algorithms that use only n function values and they have a worst case error arbitrarily close to the worst case error of A_n^{Per}. However, stability of such algorithms is not clear since then we must use huge coefficients. We leave it as an open problem for $s \geq 2$ if there are stable algorithms that use only n function values and whose worst case error are comparable to the algorithm A_n^{Per}.

We are ready to bound the worst case error of A_n^{Per}.

Theorem 12. For $k \neq 0$, we have

- for even $n \in [2, 2s]$,
$$e(A_n^{\mathrm{Per}}) \leq \frac{1}{(2\pi|k|)^{n/2}},$$
- for $n > 2s$,
$$e(A_n^{\mathrm{Per}}) \leq \left(\frac{3}{2\pi}\right)^s \frac{2}{(n - 2s + 1 + |k|)^s}.$$

□

Proof. For even $n \in [2, 2s]$, we have

$$I_k(f) - A_n^{\mathrm{Per}}(f) = I_k(f) - I_k(p_{f,n/2})$$

and (28) implies the bound on $e(A_n)$.

For $n = 2s + \ell$, we clearly have $f = f - p_{f,s} + p_{f,s}$ for all $f \in H^s$. By definition of $A_n^{\rm Per}$ and linearity of I_k we obtain

$$I_k(f) - A_n^{\rm Per}(f) \;=\; I_k(f - p_{f,s}) - A^*_{\ell+1}(f - p_{n,s}).$$

From (16) we know that

$$|I_k(f - p_{f,s}) - A^*_{\ell+1}(f - p_{f,s})| \le \left(\frac{3}{2\pi}\right)^s \frac{2}{(n - 2s + 1 + |k|)^s} \, \|f - p_{f,s}\|_{H^s}.$$

Then (27) with $j = s$ yields $\|f - p_{f,s}\|_{H^s} \le \|f\|_{H^s}$, which implies the bound on $e_n(A_n^{\rm Per})$. This completes the proof. $\square$

- Assume now that $k \in \mathbb{Z}$.

Although k is now an arbitrary integer, our emphasis will be later on $k = 0$ or, more generally, on $|k|$ small relative to n. We discuss algorithms based on Taylor's expansion and periodization for $f \in H^s$. We define the algorithm $A_n^{\rm Tay-Per}$ for all $n \in [1, s]$ and for $n = 2s + \ell$ with $\ell \in \mathbb{N}_0$.

For $n \in [1, s]$, we compute $f^{(0)}(\frac{1}{2}), \dots, f^{(n-1)}(\frac{1}{2})$ and define

$$A_n^{\rm Tay-Per}(f) = I_k(T_{f,n}),$$

where $T_{f,n}$ is Taylor's expansion of f at $\frac{1}{2}$ up to the $(n-1)$st derivative and $I_k(T_{f,n})$ is explicitly given by (31) for $k = 0$ and by (32) for $k \ne 0$. For $n = 2s + \ell$ with $\ell \in \mathbb{N}_0$, we define

$$A_n^{\rm Tay-Per}(f) = A_n^{\rm Per}(f),$$

where $A_n^{\rm Per}$ is from the previous subsection.

Clearly, the algorithm $A_n^{\rm Tay-Per}$ uses at most n evaluations of f. For $s = 1$ it uses only function values and it is defined for all n. For $s \ge 2$, it also uses derivatives of f and it is not defined for $n \in [s+1, 2s-1]$. Its weights are complex and the sum of their absolute values is uniformly bounded in n. Hence, $A_n^{\rm Tay-Per}$ is stable, and a similar remark on the approximation of derivatives by function values can be made as for the algorithm $A_n^{\rm Per}$.

The worst case error of $A_n^{\rm Tay-Per}$ can be easily bounded by (30) and Theorem 12. We summarize these bounds in the following theorem.

Theorem 13. For an arbitrary integer k, we have

- for $n \in [1, s]$,

$$e(A_n^{\rm Tay-Per}) \le \frac{1}{2^{n-1}\, n!},$$

- for $n \geq 2s$
$$e(A_n^{\mathrm{Tay-Per}}) \leq \left(\frac{3}{2\pi}\right)^s \frac{2}{(n-2s+1+|k|)^s}.$$

We now comment on Theorems 12 and 13 for a finite s. For $k=0$ and initial n, i.e., even $n \leq 2s$ or $n \leq s$, we can only apply Theorem 13. It tells us that for $n \in [1,s]$ the error bound of $A_n^{\mathrm{Tay-Per}}$ is exponentially small in n. Note that for non-zero k we can use both theorems. For the initial n and $|k|$ small relative to n, the first bound of Theorem 13 is smaller than the first bound in Theorem 12. On the other hand, for large $|k|$ relative to n, the opposite is true. Obviously for $n > 2s$, both theorems coincide and the error bound of $A_n^{\mathrm{Tay-Per}} = A_n^{\mathrm{Per}}$ is of the form

$$e(A_n^{\mathrm{Per}}) \;\leq\; \left(\frac{3}{2\pi}\right)^s \frac{2}{(n-2s+1+|k|)^s}. \tag{33}$$

The last bound yields an upper bound on the nth minimal error $e(n,k,s)$ for $n \geq 2s$. Combining this with (20) and Theorem 8 we obtain sharp lower and upper bounds on the minimal errors $e(n,k,s)$.

Theorem 14. Consider the integration problem I_k defined over the space H^s of non-periodic functions with $s \in \mathbb{N}$. Then

$$\frac{c_s}{(n+|k|)^s} \;\leq\; e(n,k,s) \;\leq\; \left(\frac{3}{2\pi}\right)^s \frac{2}{(n+|k|-2s+1)^s},$$

for all $k \in \mathbb{Z}$ and $n \geq 2s$. The positive number c_s is from Theorem 8. □

We stress that Theorem 14 presents an upper bound on the minimal errors $e(n,k,s)$ only for $n \geq 2s$, the lower bound holds for all n. The reason is that we need $2s$ function and derivatives values to periodize the function f which enables us to use the algorithm A_n^*. We do not know sharp bounds on $e(n,k,s)$ for $n \in [1,2s)$. However, we know that $e(n,k,s)$ is at most $1/(2^{n-1}n!)$ for $n \leq s$ and all k, see Theorem 13, and at most $(2\pi|k|)^{-n/2}$ for $n \leq 2s$ and all $k \neq 0$, see Theorem 12. Of course, the problem of the minimal errors $e(n,k,s)$ for initial n it is not very important as long as s is not too large.

The minimal errors $e(n,k,s)$ for the non-periodic case have a peculiar property for $s \geq 2$ and large k. Namely, for $n=0$ we obtain the initial error which is of order $|k|^{-1}$, whereas for $n \geq 2s$ it becomes of order $|k|^{-s}$. Hence, the dependence on $|k|^{-1}$ is short-lived and disappears quite quickly. For instance, take $s=2$. Then $e(n,k,s)$ is of order $|k|^{-1}$ only for $n=0$ and maybe for $n=1,2,3$, and then becomes of order $|k|^{-2}$.

We now briefly discuss the absolute and normalized error criteria for the non-periodic case. For the *absolute error criterion*, the information complexity $n^{\rm abs}(\varepsilon,k,s)$ for $\varepsilon\in(0,1)$ is defined as

$$n^{\rm abs}(\varepsilon,k,s)=\min\{\,n\mid\ e(n,k,s)\le\varepsilon\,\}\,.$$

Clearly, $n^{\rm abs}(\varepsilon,k,s)=0$ for $\varepsilon\ge e(0,k,s)$. For $\varepsilon<e(0,k,s)$ we can bound $n^{\rm abs}(\varepsilon,k,s)$ by Theorem 14. This implies the following corollary.

Corollary 15. Consider the absolute error criterion for the integration problem I_k defined over the space H^s. Let c_s be from Theorem 8.

- For $k=0$ and all $\varepsilon\in(0,1)$ we have

$$c_s^{1/s}\left(\frac{1}{\varepsilon}\right)^{1/s}\ \le\ n^{\rm abs}(\varepsilon,0,s)\ \le\ \left\lceil\left(\frac{3}{2\pi}\right)\left(\frac{2}{\varepsilon}\right)^{1/s}\right\rceil+2s-1.$$

- For $k\neq 0$ and $\varepsilon\in[\beta_{k,s}/(2\pi|k|),1)$, with $\beta_{k,s}$ from Proposition 11, we have

$$n^{\rm abs}(\varepsilon,k,s)=0,$$

  whereas for $\varepsilon\in(0,\beta_{k,s}/(2\pi|k|))$ we have

$$c_s^{1/s}\left(\frac{1}{\varepsilon}\right)^{1/s}-\ |k|\ \le\ n^{\rm abs}(\varepsilon,k,s)\ \le\ 2s\ +\ \max\left\{0,\left\lceil\left(\frac{3}{2\pi}\right)\left(\frac{2}{\varepsilon}\right)^{1/s}\right\rceil-1-|k|\right\}.$$

$\square$

Similarly as for the periodic case, this means that for the absolute error criterion the problem for the non-periodic case becomes easier for large $|k|$. However, for $k\neq0$, the condition on ε is now quite different for $s\ge2$ as compared to the periodic case, see Corollary 9. We also stress that the asymptotic behaviors of $\tilde n^{\rm abs}(\varepsilon,k,s)$ and $n^{\rm abs}(\varepsilon,k,s)$ are of order $\varepsilon^{-1/s}$ and do not depend on k.

We now turn to the *normalized error criterion* for which the information complexity $n^{\rm nor}(\varepsilon,k,s)$ for $\varepsilon\in(0,1)$ is defined as

$$\tilde n^{\rm nor}(\varepsilon,k,s)=\min\{\,n\mid\ e(n,k,s)\le\varepsilon\,e(0,k,s)\,\}\,.$$

We always have $n^{\rm nor}(\varepsilon,k,s)\ge1$. For $k=0$ we have $e(0,0,s)=1$ and there is no difference between the normalized and absolute error criteria also for the non-periodic case.

For $k\neq0$ the situation is quite different. From Theorem 14, Proposition 3 as well as the estimates of $\beta_{k,s}$, it is easy to obtain the following corollary.

Corollary 16. Consider the normalized error criterion for the integration problem I_k defined over the space H^s. Let c_s be from Theorem 8.

For all $k \neq 0$ and all $\varepsilon \in (0,1)$ we have

$$c_s^{1/s}\left(\frac{\sqrt{2}\pi|k|}{\varepsilon}\right)^{1/s} - |k| \le n^{\mathrm{nor}}(\varepsilon,k,s) \le\; 2s + \max\left\{0, \left\lceil \left(\frac{3}{2\pi}\right)\left(\frac{4\pi|k|}{\varepsilon}\right)^{1/s}\right\rceil - 1 - |k|\right\},$$

which can be written as

$$n^{\mathrm{nor}}(\varepsilon,k,s) = \Theta\left(\frac{|k|^{1/s}}{\varepsilon^{1/s}}\right) \qquad \text{as} \qquad \varepsilon \to 0. \tag{34}$$

The asymptotic expression (34) shows that for the normalized error criterion the problem becomes harder for large $|k|$ and small ε. The dependence on k is through $|k|^{1/s}$ and decreases with s. This should be compared with the periodic case, where the dependence on $|k|$ is linear. Hence, the dependence on k is the same for $s = 1$, and the periodic case is harder than the non-periodic case for $s \ge 2$ and small ε.

For fixed ε and varying $|k|$, the difference in the behavior of the information complexity in $|k|$ is even more dramatic and depends on s. Consider first $s = 1$. Then Corollary 16 yields for $\varepsilon < \sqrt{2}\pi c_s$ that

$$\lim_{|k|\to\infty} n^{\mathrm{nor}}(\varepsilon,k,s) = \infty,$$

as for the periodic case, see (17).

Assume now that $s \ge 2$. In this case, the information complexity for the non-periodic problem does not go to infinity with $|k|$ in contrast to the periodic case, see again (17). This simply follows from Corollary 16 since the second term of the maximum behaves like $\mathcal{O}(|k|^{1/s}) - |k|$ and goes to $-\infty$. Hence

$$\limsup_{|k|\to\infty} n^{\mathrm{nor}}(\varepsilon,k,s) \le 2s. \tag{35}$$

This is even true if we choose ε slowly decreasing with $|k|$, say $\varepsilon_k = |k|^{-\eta}$ for some $\eta \in (0, s-1)$. Indeed, then $|k|/\varepsilon_k = |k|^{1+\eta}$ and $\mathcal{O}(|k|^{(1+\eta)/s}) - |k|$ still goes to $-\infty$ and (35) is again valid. This discussion can be summarized as follows.

Corollary 17. For the non-periodic case and the normalized error criterion

- for $s \ge 1$, oscillatory integration becomes harder in $|k|$ asymptotically in ε,
- for $s = 1$ and fixed small ε, oscillatory integration becomes harder in $|k|$,
- for $s \ge 2$ and fixed ε or even for $\varepsilon^{-1} = \mathcal{O}(|k|^{\eta})$ with $\eta \in (0, s-1)$, oscillatory integration becomes easy since $n^{\mathrm{nor}}(\varepsilon,k,s)$ is at most $2s$ for large $|k|$.

5 The case of $s = \infty$

We briefly discuss the space H^∞ which is defined as

$$H^\infty = \{ f \in C^\infty([0,1]) \mid \sum_{\ell=0}^{\infty} \|f^{(\ell)}\|_{L_2}^2 < \infty \},$$

where $\|f^{(\ell)}\|_{L_2}$ denotes the $L_2 = L_2([0,1])$ norm of $f^{(\ell)}$. Note that H^∞ consists of infinitely many times differentiable functions. In particular, all polynomials belong to H^∞ but $e_h(x) = \exp(2\pi i h x)$ belongs to H^∞ iff $h = 0$.

We equip the space H^∞ with the two inner products

$$\begin{aligned}
\langle f, g \rangle_\infty &= \sum_{\ell=0}^{\infty} \left\langle f^{(\ell)}, 1 \right\rangle_0 \overline{\left\langle g^{(\ell)}, 1 \right\rangle_0}, \\
\langle f, g \rangle_{\infty,*} &= \sum_{\ell=0}^{\infty} \left\langle f^{(\ell)}, g^{(\ell)} \right\rangle_0 .
\end{aligned}$$

As for a finite s, the norms generated by theses inner products are closely related since we have

$$\tfrac{12}{13} \, \|f\|_{H_*^\infty} \le \|f\|_{H^\infty} \le \|f\|_{H_*^\infty} \quad \text{for all} \quad f \in H^\infty,$$

see the appendix. This means that it is enough to consider only one of these inner product and, as before, we choose $\langle \cdot, \cdot \rangle_\infty$ for simplicity of the analysis.

Proposition 18. Polynomials are dense in H^∞, i.e., for any $f \in H^\infty$ and any positive ε there is a polynomial p such that

$$\|f - p\|_{H^\infty} \le \varepsilon.$$

Proof. We begin by showing that for an absolutely continuous function g for which $g' \in L_2([0,1])$ we have

$$\|g - g(\tfrac12)\|_{L_2} \le \tfrac12 \, \|g'\|_{L_2}. \tag{36}$$

Indeed, $g(x) - g(\frac12) = \int_{1/2}^{x} g'(t)\,\mathrm{d}t$ and

$$|g(x) - g(\tfrac12)| \le \left| \int_{1/2}^{x} |g'(t)| \,\mathrm{d}t \right| \le \left| \int_{1/2}^{x} \mathrm{d}t \right|^{1/2} \left| \int_{1/2}^{x} |g'(t)|^2 \,\mathrm{d}t \right|^{1/2} \le |x - \tfrac12|^{1/2} \, \|g'\|_{L_2}.$$

Hence,

$$\|g - g(\tfrac12)\|_{L_2} \le \left(\int_0^1 |x - \tfrac12| \,\mathrm{d}x \right)^{1/2} \|g'\|_{L_2} = \tfrac12 \, \|g'\|_{L_2},$$

as claimed.

Take now an arbitrary $f \in H^\infty$. For any positive δ there exists $\ell^* = \ell^*(f,\delta) \in \mathbb{N}$ such that

$$\sum_{\ell=\ell^*}^{\infty} \|f^{(\ell)}\|_{L_2}^2 \le \delta^2.$$

In particular, $\|f^{(\ell^*)}\|_{L_2} \le \delta$. Taking $g' = f^{(\ell^*)}$ we conclude from (36) that

$$\|f^{(\ell^*-1)} - f^{(\ell^*-1)}(\tfrac{1}{2})\|_{L_2} \le \tfrac{1}{2}\,\delta.$$

For $\ell^* \ge 2$, we take $g' = f^{(\ell^*-1)} - f^{(\ell^*-1)}(\frac{1}{2})$ and we have again from (36)

$$\|f^{(\ell^*-2)} - f^{(\ell^*-1)}(\tfrac{1}{2})(\cdot - \tfrac{1}{2}) - f^{(\ell^*-2)}(\tfrac{1}{2})\|_{L_2} \le \tfrac{1}{4}\,\delta.$$

Repeating this procedure we conclude that for

$$p(x) = f(\tfrac{1}{2}) + f^{(\prime)}(\tfrac{1}{2})(x-\tfrac{1}{2}) + \cdots + \frac{f^{(\ell^*-1)}(\frac{1}{2})}{(\ell^*-1)!}(x-\tfrac{1}{2})^{\ell^*-1}$$

we have

$$\|f^{(\ell)} - p^{(\ell)}\|_{L_2} \le 2^{\ell-\ell^*}\,\delta \qquad \text{for all} \qquad j = 0,1,\dots,\ell^*-1.$$

Hence,

$$\begin{aligned}
\|f-p\|_{H^\infty}^2 &\le \sum_{\ell=0}^{\infty} \|f^{(\ell)} - p^{(\ell)}\|_{L_2}^2 = \sum_{\ell=0}^{\ell^*-1} \|f^{(\ell)} - p^{(\ell)}\|_{L_2}^2 \;+\; \sum_{\ell=\ell^*}^{\infty} \|f^{(\ell)} - p^{(\ell)}\|_{L_2}^2 \\
&\le \delta^2 \sum_{\ell=0}^{\ell^*-1} 4^{\ell-\ell^*} \;+\; \delta^2 = \tfrac{4}{3}\,\delta^2.
\end{aligned}$$

Taking $\delta = \sqrt{3/4}\,\varepsilon$, Proposition 18 is proved. $\square$

It is easy to see that that the periodic subspace

$$\widetilde{H}^\infty = \{\, f \in H^\infty \mid f^{(\ell)}(0) = f^{(\ell)}(1) \;\; \text{for} \;\; \ell \in \mathbb{N}_0\}$$

consists only of constant functions. Indeed, since $\|I_k\|_{\widetilde{H}^\infty \to \mathbb{C}} \le \|I_k\|_{\widetilde{H}^s \to \mathbb{C}}$ for all $s \in \mathbb{N}$ and for $k \ne 0$ we have $\|I_k\|_{\widetilde{H}^s \to \mathbb{C}} = (2\pi|k|)^{-s}$, we conclude that $I_k = 0$ for all $k \ne 0$. This means that $f \in \widetilde{H}^\infty$ implies that $f =$ constant, as claimed. It is also easy to check that the reproducing kernel of $\widetilde{H}^\infty$ is $\widetilde{K}_\infty(x,t) = 1$.

Let $\tilde{e}(n,k,\infty)$ be the minimal errors for $\widetilde{H}^\infty$. Then $\tilde{e}(0,0,\infty)=1$ and $\tilde{e}(n,0,\infty)=0$ for all $n\ge 1$, whereas $\tilde{e}(n,k,\infty)=0$ for all $n\ge 0$ and $k\ne 0$.

This means that the periodic case is trivial and cannot be used as a tool for the non-periodic case. That is why our lower bound on $\tilde{e}(n,k,s)$ which was quite useful for a finite s is meaningless for $s=\infty$. In fact, the problem of non-trivial lower bounds for H^∞ is open.

Proposition 18, together with (23), shows that the set of normalized Bernoulli polynomials $\{B_j^*\}_{j=0,1,\dots}$ is a complete orthonormal basis of H^∞ and therefore the reproducing kernel K_∞ is given by

$$K_\infty(x,t)=\sum_{j=0}^{\infty} B_j^*(x)\,B_j^*(t) \qquad \text{for all} \qquad x,t\in[0,1].$$

We now present some upper error bounds on the minimal errors $e(n,k,\infty)$ for H^∞. In fact, we derived the upper bounds in Theorems 12 and 13 in such a way that they can be used even for $s=\infty$.

We start with the initial error. For $k=0$, the representer of I_0 is 1 and

$$e(0,0,\infty)=1,$$

whereas for $k\ne 0$, the proof of Proposition 11 can be modified for $s=\infty$ and yields that the representer of I_k is

$$h_{k,\infty}(x)=-\sum_{\ell=1}^{\infty}(-1)^\ell\,(2\pi i k)^{-\ell}\,B_\ell^*(x)$$

and

$$e(0,k,\infty)=\frac{\beta_{k,\infty}}{2\pi|k|}$$

with

$$\beta_{k,\infty}=\left(\frac{4\pi^2k^2}{4\pi^2k^2-1}\right)^{1/2}\in[1,1.013].$$

For $k=0$ and all $n\ge 1$, we can apply the first error bound in Theorem 13 which states that

$$e(n,0,\infty)\le\frac{1}{2^{n-1}\,n!},$$

which is super exponentially small in n.

For $k\ne 0$ and all even n, we apply the first error bounds in Theorems 12 and 13 which state that

$$e(n,k,\infty)\le\min\left(\frac{1}{2^{n-1}\,n!},\ \frac{1}{(2\pi|k|)^{n/2}}\right).$$

Note that by Stirling's approximation we have

$$\frac{1}{2^{n-1}\,n!} \;\le\; 2\left(\frac{\mathrm{e}}{2n}\right)^{n}.$$

It is easy to check that the right hand side is smaller than ε iff

$$n\big(\ln(2n)-1\big) \ge \ln(2/\varepsilon)$$

which holds, in particular, if $n \ge 2\ln(\varepsilon^{-1})/\ln\ln(\varepsilon^{-1})$ and $\varepsilon < \mathrm{e}^{-\mathrm{e}} = 0.135\ldots$.

These upper error bounds can be used to estimate the information complexities n^{abs} and n^{nor} for $\varepsilon < \mathrm{e}^{-\mathrm{e}}$. For $k=0$, we have

$$n^{\mathrm{abs}}(\varepsilon,0,\infty) = n^{\mathrm{nor}}(\varepsilon,0,\infty) \;\le\; \left\lceil 2\,\frac{\ln\,\varepsilon^{-1}}{\ln\,\ln\,\varepsilon^{-1}}\right\rceil.$$

For $k \ne 0$, we have

$$n^{\mathrm{abs}}(\varepsilon,k,\infty) \le \left\lceil 2\,\min\left\{\frac{\ln\,\varepsilon^{-1}}{\ln\,\ln\,\varepsilon^{-1}},\ \frac{\ln\,\varepsilon^{-1}}{\ln\,(2\pi|k|)}\right\}\right\rceil,$$

and

$$n^{\mathrm{nor}}(\varepsilon,k,\infty) \le \left\lceil 2\,\min\left\{\frac{\ln\,\varepsilon^{-1}+\ln(2\pi|k|)}{\ln\,(\ln\,\varepsilon^{-1}+\ln(2\pi|k|))},\ \frac{\ln\,\varepsilon^{-1}}{\ln\,(2\pi|k|)}+1\right\}\right\rceil.$$

These estimates are valid for all $\varepsilon < \mathrm{e}^{-\mathrm{e}}$. Note that asymptotically, when ε tends to zero, all information complexity are upper bounded by roughly $\ln(\varepsilon^{-1})/\ln(\ln(\varepsilon^{-1}))$ independently of k.

6 Appendix

We now prove (4) which shows the embeddings constants between the space H^s equipped with the norm $\|\cdot\|_{H^s}$ given by (2) and $\|\cdot\|_{H^s_*}$ given by (3). Since $|\left\langle f^{(\ell)},1\right\rangle_0|^2 \le \|f^{(\ell)}\|_{L_2}^2$ we clearly have $\|f\|_{H^s} \le \|f\|_{H^s_*}$ for all $f \in H^s$.

To obtain the other estimate, we consider $f \in H^s$ which can can be written as

$$f = \sum_{j=0}^{s}\left\langle f, B_j^*\right\rangle_s B_j^* \;+\; \sum_{h\in\mathbb{Z}\setminus 0}\left\langle f, e_h\right\rangle_s e_h$$

since the normalized Bernoulli polynomials B_j^* and $e_h^* = \exp(2\pi i h\cdot)/(2\pi|h|)^s$ are an orthonormal basis of H^s with respect to $\langle\cdot,\cdot\rangle_s$. Let $\{b_j\}_{j\in\mathbb{N}}$ be some ordering of this orthonormal basis. Then, clearly, we have

$$\begin{aligned}
\|f\|_{H_*^s}^2 &= \left\|\sum_{j\in\mathbb{N}} \langle f, b_j\rangle_s \, b_j\right\|_{H_*^s}^2 = \sum_{j,m\in\mathbb{N}} \langle f, b_j\rangle_s \, \overline{\langle f, b_m\rangle_s} \, \langle b_j, b_m\rangle_{s,*} \\
&\le \sum_{j,m\in\mathbb{N}} \left|\langle f, b_j\rangle_s\right|^2 \left|\langle b_j, b_m\rangle_{s,*}\right| \\
&\le \left(\max_{m\in\mathbb{N}} \left\{\sum_{j\in\mathbb{N}} \left|\langle b_j, b_m\rangle_{s,*}\right|\right\}\right) \left(\sum_{j\in\mathbb{N}} \left|\langle f, b_j\rangle_s\right|^2\right) \\
&= \left(\max_{m\in\mathbb{N}} \left\{\sum_{j\in\mathbb{N}} \left|\langle b_j, b_m\rangle_{s,*}\right|\right\}\right) \|f\|_{H^s}^2 \\
&=: M_s \, \|f\|_{H^s}^2 .
\end{aligned} \tag{37}$$

To bound M_s we estimate $|\langle b_j, b_m\rangle_{s,*}|$ for all possible $b_j, b_m \in \{B_j^*, e_h^*\}_{j=0,1,\ldots,s,\, h\in\mathbb{Z}\setminus\{0\}}$.

We start with the case where both b_j, b_m are in $\{e_h^*\}_{h\in\mathbb{Z}\setminus 0}$. This case is easy since $\{e_h^*\}_{h\in\mathbb{Z}}$ is also an orthogonal basis in $\widetilde{H}^s$ with the inner product $\langle\cdot,\cdot\rangle_{s,*}$. We have for $h \in \mathbb{Z}\setminus 0$ and $\ell \neq h$ that

$$\langle e_h^*, e_\ell^*\rangle_{s,*} = 0 \quad \text{and} \quad \|e_h^*\|_{H_*^s} = \frac{1}{(2\pi|h|)^s} \left(\sum_{\ell=0}^{s} (2\pi|h|)^{2\ell}\right)^{1/2}.$$

Hence

$$\left|\langle e_h^*, e_h^*\rangle_{s,*}\right| = \|e_h^*\|_{H_*^s}^2 \le \frac{1}{1-(2\pi|h|)^{-2}} \le \frac{4\pi^2}{4\pi^2-1}. \tag{38}$$

To treat the case where b_j, b_m are in $\{B_j^*\}_{j=0.1,\ldots s}$, we need the following known properties of the normalized Bernoulli polynomials

$$\begin{aligned}
\langle B_m^*, B_0^*\rangle_0 &= 1 \;\text{ for } m=0 \text{ and } \;0 \text{ for } m\ge 1, \\
\left\langle B_m^*, B_j^*\right\rangle_0 &= (-1)^{\min\{m,j\}-1} B_{m+j}^*(0), \quad \text{for all } m,j\ge 1 \\
B_0^*(0) &= 1, \\
B_{2m}^*(0) &= \frac{2(-1)^{m+1}}{(2\pi)^{2m}} \zeta(2m) \quad \text{for all } m\ge 1, \\
B_{2m+1}^*(0) &= 0 \quad \text{for all } m\ge 1.
\end{aligned}$$

Here, as always, ζ is the zeta Riemann function.

From these properties for $m \in [0,s]$ we conclude

$$\|B_m^*\|_{H_*^s}^2 = \sum_{\ell=0}^{s} \|[B_m^*]^{(\ell)}\|_{L_2}^2 = \sum_{\ell=0}^{m} \|B_{m-\ell}^*\|_{L_2}^2 = 1 + 2\sum_{\ell=1}^{m} \frac{\zeta(2\ell)}{(2\pi)^{2\ell}}.$$

Hence

$$\|B_m^*\|_{H_*^s}^2 \le 1 + \frac{2\zeta(2)}{(2\pi)^2} + \frac{2\zeta(4)}{4\pi^2(4\pi^2-1)}.$$

Since $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we conclude that

$$\|B_m^*\|_{H_*^s}^2 \le 1 + \tfrac{1}{12} + \frac{\pi^2}{180(4\pi^2-1)}. \tag{39}$$

We now consider $\left\langle B_m^*, B_j^* \right\rangle_{s,*}$ for all $m,j \in [0,s]$ and $m \neq j$. Let $m' = \max\{m,j\}$ and $j' = \min\{m,j\}$. Furthermore, let $\kappa_{m'-j'} = 0$ for odd $m'-j'$ and $\kappa_{m'-j'} = 1$ for even $m'-j'$. Then

$$\langle B_m^*, B_0^* \rangle_{s,*} = \langle B_m^*, B_0^* \rangle_0 = \delta_{m,0},$$

whereas for $m,j \in [1,s]$ and $m \neq j$ we have

$$\begin{aligned}
\left\langle B_m^*, B_j^* \right\rangle_{s,*} &= \sum_{\ell=0}^{j'-1} \left\langle B_{m-\ell}^*, B_{j-\ell}^* \right\rangle_0 = \sum_{\ell=0}^{j'-1} (-1)^{j'-\ell-1} B_{m+j-2\ell}^*(0) \\
&= 2\kappa_{m'-j'}(-1)^{(m'-j')/2} \sum_{\ell=0}^{j'-1} \frac{\zeta(m+j-2\ell)}{(2\pi)^{m+j-2\ell}} \\
&= 2\kappa_{m'-j'}(-1)^{(m'-j')/2} \sum_{\ell=1}^{j'} \frac{\zeta(m'-j'+2\ell)}{(2\pi)^{m'-j'+2\ell}}.
\end{aligned}$$

Note that the smallest argument of ζ for even $m'-j'$ is 4. Therefore

$$|\left\langle B_m^*, B_j^* \right\rangle_{s,*}| \le 2\zeta(4)\,\frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}}\,\sum_{\ell=1}^{\infty} \frac{1}{(2\pi)^{2l}} \le \frac{2\zeta(4)}{4\pi^2-1}\,\frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}}.$$

From this we have

$$\sum_{m=0}^{s} |\left\langle B_m^*, B_j^* \right\rangle_{s,*}| \le \|B_j^*\|_{H_*^s}^2 + \frac{2\zeta(4)}{4\pi^2-1}\,\sum_{\substack{m=0\\ m\neq j}}^{s} \frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}}.$$

The non-zero terms correspond to even $m' - j'$ and each non-zero term $\kappa_{m'-j'}/(2\pi)^{m'-j'}$ may appear at most twice. Therefore

$$\sum_{m=0}^{s} |\left\langle B_m^*, B_j^* \right\rangle_{s,*}| \leq \|B_j^*\|_{H_*^s}^2 + \frac{4\zeta(4)}{4\pi^2 - 1} \sum_{\ell=0}^{\infty} \frac{1}{(2\pi)^{2\ell}} = \|B_j^*\|_{H_*^s}^2 + \frac{4\pi^4}{90(4\pi^2-1)^2}. \tag{40}$$

We now consider the coefficients $\left\langle B_j^*, e_h^* \right\rangle_s$ for $j = 0, 1, \ldots, s$ and $h \in \mathbb{Z} \setminus 0$. We start by showing that we have

$$\int_0^1 B_j^*(x)\, e^{-2\pi i\, hx}\, \mathrm{d}x = \begin{cases} 0 & \text{for } \; j = 0, \\ \frac{-1}{(2\pi i\, h)^j} & \text{for } \; j \geq 1. \end{cases}$$

For $j = 0$, it is zero since the integral of e_h^* is zero for $h \neq 0$.

We now use induction on j. For $j = 1$, we have $B_1^*(x) = x - \frac{1}{2}$ and using integration by parts we get

$$\int_0^1 \left(x - \tfrac{1}{2}\right)\, e^{-2\pi i\, hx}\, \mathrm{d}x = \frac{-1}{2\pi i\, h} \int_0^1 \left(x - \tfrac{1}{2}\right)\, \mathrm{d}\, e^{-2\pi i\, hx} = \frac{-1}{2\pi i\, h} \left(1 - \int_0^1 e^{-2\pi i\, hx}\, \mathrm{d}x\right) = \frac{-1}{2\pi i\, h},$$

as claimed.

For $j > 1$, we again use integration by parts, and the property of Bernoulli polynomials (24) and (25) to obtain

$$\begin{aligned} \int_0^1 B_j^*(x)\, e^{-2\pi i\, hx}\, \mathrm{d}x &= \frac{-1}{2\pi i\, h} \int_0^1 B_j^*(x)\, \mathrm{d}\, e^{-2\pi i\, hx} = \frac{1}{2\pi i\, h} \int_0^1 [B_j^*]'(x)\, e^{-2\pi i\, hx}\, \mathrm{d}x \\ &= \frac{1}{2\pi i\, h} \int_0^1 B_{j-1}^*(x)\, e^{-2\pi i\;\, hx}\, \mathrm{d}x = \frac{-1}{(2\pi i\, h)^j}, \end{aligned}$$

as claimed.

From this we conclude that for $\ell \leq j$ and $j \neq 0$,

$$\left\langle [B_j^*]^{(\ell)}, [e^{2\pi i\, h\cdot}]^{(\ell)} \right\rangle_0 = (-2\pi i\, h)^\ell \left\langle B_{j-\ell}^*, e^{2\pi i\, h\cdot} \right\rangle_0 = -(2\pi h)^{2\ell - j}\, i^{-j}.$$

Clearly, for $j = 0$ we have $\left\langle [B_j^*]^{(\ell)}, [e^{2\pi i\, h\cdot}]^{(\ell)} \right\rangle_0 = 0$.

Then for $h \neq 0$ we have

$$\begin{aligned} \left\langle B_j^*, e_h^* \right\rangle_{s,*} &= \sum_{\ell=0}^{s} \left\langle [B_j^*]^{(\ell)}, [e_h^*]^{(\ell)} \right\rangle_0 = \frac{1}{(2\pi |h|)^s} \sum_{\ell=0}^{j-1} \left\langle [B_j^*]^{(\ell)}, [e^{2\pi i\, h\cdot}]^{(\ell)} \right\rangle_0 \\ &= \begin{cases} 0 & \text{for } \; j = 0, \\ \frac{-1}{(2\pi |h|)^s (2\pi i h)^j} \sum_{\ell=0}^{j-1} (2\pi h)^{2\ell} & \text{for } \; j \geq 1. \end{cases} \end{aligned}$$

This allows us to compute $\left|\left\langle B_j^*, e_h^*\right\rangle_{s,*}\right|$ for $h \neq 0$ and $j = 0, \ldots, s$. For $j = 0$ it is zero, and for $j \geq 1$ we have

$$\left|\left\langle B_j^*, e_h^*\right\rangle_{s,*}\right| = \frac{1}{(2\pi|h|)^{s+j}} \frac{(2\pi h)^{2j} - 1}{(2\pi h)^2 - 1} < \frac{(2\pi|h|)^{j-s}}{(2\pi h)^2 - 1} \leq \frac{4\pi^2}{4\pi^2 - 1} \frac{1}{(2\pi|h|)^{s-j+2}}. \qquad (41)$$

We now are ready to bound M_s from (37). For this we define

$$M_{s,1} := \max_{h \in \mathbb{Z} \setminus 0} \left\{ \left|\langle e_h^*, e_h^* \rangle_{s,*}\right| + \sum_{j=0}^{s} \left|\left\langle B_j^*, e_h^*\right\rangle_{s,*}\right| \right\}$$

and

$$M_{s,2} := \max_{j=0,\ldots,s} \left\{ \sum_{h \in \mathbb{Z} \setminus 0} \left|\left\langle B_j^*, e_h^*\right\rangle_{s,*}\right| + \sum_{m=0}^{s} \left|\left\langle B_m^*, B_j^*\right\rangle_{s,*}\right| \right\}$$

such that $M_s = \max\{M_{s,1}, M_{s,2}\}$.

Clearly, M_s is strictly larger than one since the term inside of the maximum of $M_{s,2}$ corresponding to $j = 0$ equals one. We now show that M_s is close to one.

From (38) and (41) we get

$$M_{s,1} \leq \frac{4\pi^2}{4\pi^2 - 1} \left(1 + \sum_{j=1}^{s} \frac{1}{(2\pi)^{s-j+2}} \right) \leq \frac{4\pi^2}{4\pi^2 - 1} \left(1 + \frac{1}{(2\pi)^2 - 2\pi} \right) \leq 1.057.$$

We now bound $M_{s,2}$. We use (41) to bound the first sum in $M_{s,2}$,

$$\begin{aligned} \sum_{h \in \mathbb{Z} \setminus 0} \left|\left\langle B_j^*, e_h^*\right\rangle_{s,*}\right| &\leq \frac{4\pi^2}{4\pi^2 - 1} \sum_{h \in \mathbb{Z} \setminus 0} \frac{1}{(2\pi|h|)^{s-j+2}} = \frac{8\pi^2}{4\pi^2 - 1} \frac{\zeta(s-j+2)}{(2\pi)^{s-j+2}} \\ &\leq \frac{8\pi^2}{4\pi^2 - 1} \frac{\zeta(2)}{(2\pi)^2} = \frac{\pi^2}{3(4\pi^2 - 1)} \leq 0.0855. \end{aligned}$$

Using (39) and (40) we can bound the second sum in $M_{s,2}$ by

$$\sum_{m=0}^{s} \left|\left\langle B_m^*, B_j^*\right\rangle_{s,*}\right| \leq \tfrac{13}{12} + \frac{\pi^2}{180(4\pi^2 - 1)} + \frac{4\pi^4}{90(4\pi^2 - 1)^2} \leq 1.0877.$$

This shows that $M_{s,2} \leq 1.1732$ and, consequently, $M_s \leq 1.1732$ and $\sqrt{M_s} \leq 1.0832 \leq \frac{13}{12}$. From (37) we finally obtain

$$\|f\|_{H_*^s} \leq \sqrt{M_s}\, \|f\|_{H^s} \leq \tfrac{13}{12}\, \|f\|_{H^s}$$

for all $f \in H^s$, as claimed

References

[1] N. S. Bakhvalov, On the optimality of linear methods for operator approximation in convex classes of functions, *USSR Comp. Math. Math. Phys.* **11**, 244–249, 1971.

[2] A. Berlinet and C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer, Boston, 2004.

[3] P. Craven and G. Wahba, Smoothing noisy data with spline functions, *Numer. Math.* **31**, 377–403, 1979.

[4] V. Domínguez, I. G. Graham and V. P. Smyshlyaev, Stability and error estimates for Filon-Clenshaw-Curtis rules for highly oscillatory integrals, IMA Journal of Numerical Analysis **31**, 1253–1280, 2011.

[5] D. Huybrechs and S. Olver, Highly oscillatory quadrature, *London Math. Soc. Lecture Note Ser.* **366**, Cambridge, 2009.

[6] A. Iserles and S. Nørsett, Efficient quadrature of highly oscillatory integrals using derivatives, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461**, 1383–1399, 2005.

[7] Y. Kametaka, H. Yamagishi, K. Watanabe, A. Nagai and K. Takemura, Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality, Scientiae Mathematicae Japonicae Online **e-2007**, 63–89, 2006.

[8] J. M. Melenk, On the convergence of Filon quadrature, J. Comput. Applied Math. **234**, 1692–1701, 2010.

[9] E. Novak, *Deterministic and Stochastic Error Bounds in Numerical Analysis*, LNiM **1349**, Springer-Verlag, Berlin, 1988.

[10] E. Novak and H. Woźniakowski, *Tractability of Multivariate Problems*, Volume I: Linear Information, European Math. Soc. Publ. House, Zürich, 2008.

[11] S. Olver, Numerical Approximation of Highly Oscillatory Integrals, dissertation, University of Cambridge, 2008.

[12] S. A. Smolyak, On optimal restoration of functions and functionals of them, (in Russia), Candidate Dissertation, Moscow State University, 1965.

[13] J. F. Traub, G. W. Wasilkowski and H. Woźniakowski, *Information-Based Complexity*, Academic Press, 1988.

[14] G. Wahba, *Spline models for observational data*, CBMS-NSF Regional Conference Series in Applied Mathematics, 59, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

Preprint Series DFG-SPP 1324

http://www.dfg-spp1324.de

Reports

[1] R. Ramlau, G. Teschke, and M. Zhariy. A Compressive Landweber Iteration for Solving Ill-Posed Inverse Problems. Preprint 1, DFG-SPP 1324, September 2008.

[2] G. Plonka. The Easy Path Wavelet Transform: A New Adaptive Wavelet Transform for Sparse Representation of Two-dimensional Data. Preprint 2, DFG-SPP 1324, September 2008.

[3] E. Novak and H. Woźniakowski. Optimal Order of Convergence and (In-) Tractability of Multivariate Approximation of Smooth Functions. Preprint 3, DFG-SPP 1324, October 2008.

[4] M. Espig, L. Grasedyck, and W. Hackbusch. Black Box Low Tensor Rank Approximation Using Fibre-Crosses. Preprint 4, DFG-SPP 1324, October 2008.

[5] T. Bonesky, S. Dahlke, P. Maass, and T. Raasch. Adaptive Wavelet Methods and Sparsity Reconstruction for Inverse Heat Conduction Problems. Preprint 5, DFG-SPP 1324, January 2009.

[6] E. Novak and H. Woźniakowski. Approximation of Infinitely Differentiable Multivariate Functions Is Intractable. Preprint 6, DFG-SPP 1324, January 2009.

[7] J. Ma and G. Plonka. A Review of Curvelets and Recent Applications. Preprint 7, DFG-SPP 1324, February 2009.

[8] L. Denis, D. A. Lorenz, and D. Trede. Greedy Solution of Ill-Posed Problems: Error Bounds and Exact Inversion. Preprint 8, DFG-SPP 1324, April 2009.

[9] U. Friedrich. A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs. Preprint 9, DFG-SPP 1324, April 2009.

[10] K. Bredies and D. A. Lorenz. Minimization of Non-smooth, Non-convex Functionals by Iterative Thresholding. Preprint 10, DFG-SPP 1324, April 2009.

[11] K. Bredies and D. A. Lorenz. Regularization with Non-convex Separable Constraints. Preprint 11, DFG-SPP 1324, April 2009.

[12] M. Döhler, S. Kunis, and D. Potts. Nonequispaced Hyperbolic Cross Fast Fourier Transform. Preprint 12, DFG-SPP 1324, April 2009.

[13] C. Bender. Dual Pricing of Multi-Exercise Options under Volume Constraints. Preprint 13, DFG-SPP 1324, April 2009.

[14] T. Müller-Gronbach and K. Ritter. Variable Subspace Sampling and Multi-level Algorithms. Preprint 14, DFG-SPP 1324, May 2009.

[15] G. Plonka, S. Tenorth, and A. Iske. Optimally Sparse Image Representation by the Easy Path Wavelet Transform. Preprint 15, DFG-SPP 1324, May 2009.

[16] S. Dahlke, E. Novak, and W. Sickel. Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings IV: Errors in L_2 and Other Norms. Preprint 16, DFG-SPP 1324, June 2009.

[17] B. Jin, T. Khan, P. Maass, and M. Pidcock. Function Spaces and Optimal Currents in Impedance Tomography. Preprint 17, DFG-SPP 1324, June 2009.

[18] G. Plonka and J. Ma. Curvelet-Wavelet Regularized Split Bregman Iteration for Compressed Sensing. Preprint 18, DFG-SPP 1324, June 2009.

[19] G. Teschke and C. Borries. Accelerated Projected Steepest Descent Method for Nonlinear Inverse Problems with Sparsity Constraints. Preprint 19, DFG-SPP 1324, July 2009.

[20] L. Grasedyck. Hierarchical Singular Value Decomposition of Tensors. Preprint 20, DFG-SPP 1324, July 2009.

[21] D. Rudolf. Error Bounds for Computing the Expectation by Markov Chain Monte Carlo. Preprint 21, DFG-SPP 1324, July 2009.

[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.

[23] F.J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter. Multi-level Monte Carlo Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 23, DFG-SPP 1324, August 2009.

[24] S. Dereich and F. Heidenreich. A Multilevel Monte Carlo Algorithm for Lévy Driven Stochastic Differential Equations. Preprint 24, DFG-SPP 1324, August 2009.

[25] S. Dahlke, M. Fornasier, and T. Raasch. Multilevel Preconditioning for Adaptive Sparse Optimization. Preprint 25, DFG-SPP 1324, August 2009.

[26] S. Dereich. Multilevel Monte Carlo Algorithms for Lévy-driven SDEs with Gaussian Correction. Preprint 26, DFG-SPP 1324, August 2009.

[27] G. Plonka, S. Tenorth, and D. Roşca. A New Hybrid Method for Image Approximation using the Easy Path Wavelet Transform. Preprint 27, DFG-SPP 1324, October 2009.

[28] O. Koch and C. Lubich. Dynamical Low-rank Approximation of Tensors. Preprint 28, DFG-SPP 1324, November 2009.

[29] E. Faou, V. Gradinaru, and C. Lubich. Computing Semi-classical Quantum Dynamics with Hagedorn Wavepackets. Preprint 29, DFG-SPP 1324, November 2009.

[30] D. Conte and C. Lubich. An Error Analysis of the Multi-configuration Time-dependent Hartree Method of Quantum Dynamics. Preprint 30, DFG-SPP 1324, November 2009.

[31] C. E. Powell and E. Ullmann. Preconditioning Stochastic Galerkin Saddle Point Problems. Preprint 31, DFG-SPP 1324, November 2009.

[32] O. G. Ernst and E. Ullmann. Stochastic Galerkin Matrices. Preprint 32, DFG-SPP 1324, November 2009.

[33] F. Lindner and R. L. Schilling. Weak Order for the Discretization of the Stochastic Heat Equation Driven by Impulsive Noise. Preprint 33, DFG-SPP 1324, November 2009.

[34] L. Kämmerer and S. Kunis. On the Stability of the Hyperbolic Cross Discrete Fourier Transform. Preprint 34, DFG-SPP 1324, December 2009.

[35] P. Cerejeiras, M. Ferreira, U. Kähler, and G. Teschke. Inversion of the noisy Radon transform on $SO(3)$ by Gabor frames and sparse recovery principles. Preprint 35, DFG-SPP 1324, January 2010.

[36] T. Jahnke and T. Udrescu. Solving Chemical Master Equations by Adaptive Wavelet Compression. Preprint 36, DFG-SPP 1324, January 2010.

[37] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Irregular Shearlet Frames: Geometry and Approximation Properties. Preprint 37, DFG-SPP 1324, February 2010.

[38] G. Kutyniok and W.-Q Lim. Compactly Supported Shearlets are Optimally Sparse. Preprint 38, DFG-SPP 1324, February 2010.

[39] M. Hansen and W. Sickel. Best m-Term Approximation and Tensor Products of Sobolev and Besov Spaces – the Case of Non-compact Embeddings. Preprint 39, DFG-SPP 1324, March 2010.

[40] B. Niu, F.J. Hickernell, T. Müller-Gronbach, and K. Ritter. Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 40, DFG-SPP 1324, March 2010.

[41] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Construction of Compactly Supported Shearlet Frames. Preprint 41, DFG-SPP 1324, March 2010.

[42] C. Bender and J. Steiner. Error Criteria for Numerical Solutions of Backward SDEs. Preprint 42, DFG-SPP 1324, April 2010.

[43] L. Grasedyck. Polynomial Approximation in Hierarchical Tucker Format by Vector-Tensorization. Preprint 43, DFG-SPP 1324, April 2010.

[44] M. Hansen und W. Sickel. Best m-Term Approximation and Sobolev-Besov Spaces of Dominating Mixed Smoothness - the Case of Compact Embeddings. Preprint 44, DFG-SPP 1324, April 2010.

[45] P. Binev, W. Dahmen, and P. Lamby. Fast High-Dimensional Approximation with Sparse Occupancy Trees. Preprint 45, DFG-SPP 1324, May 2010.

[46] J. Ballani and L. Grasedyck. A Projection Method to Solve Linear Systems in Tensor Format. Preprint 46, DFG-SPP 1324, May 2010.

[47] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence Rates for Greedy Algorithms in Reduced Basis Methods. Preprint 47, DFG-SPP 1324, May 2010.

[48] S. Kestler and K. Urban. Adaptive Wavelet Methods on Unbounded Domains. Preprint 48, DFG-SPP 1324, June 2010.

[49] H. Yserentant. The Mixed Regularity of Electronic Wave Functions Multiplied by Explicit Correlation Factors. Preprint 49, DFG-SPP 1324, June 2010.

[50] H. Yserentant. On the Complexity of the Electronic Schrödinger Equation. Preprint 50, DFG-SPP 1324, June 2010.

[51] M. Guillemard and A. Iske. Curvature Analysis of Frequency Modulated Manifolds in Dimensionality Reduction. Preprint 51, DFG-SPP 1324, June 2010.

[52] E. Herrholz and G. Teschke. Compressive Sensing Principles and Iterative Sparse Recovery for Inverse and Ill-Posed Problems. Preprint 52, DFG-SPP 1324, July 2010.

[53] L. Kämmerer, S. Kunis, and D. Potts. Interpolation Lattices for Hyperbolic Cross Trigonometric Polynomials. Preprint 53, DFG-SPP 1324, July 2010.

[54] G. Kutyniok and W.-Q Lim. Shearlets on Bounded Domains. Preprint 54, DFG-SPP 1324, July 2010.

[55] A. Zeiser. Wavelet Approximation in Weighted Sobolev Spaces of Mixed Order with Applications to the Electronic Schrödinger Equation. Preprint 55, DFG-SPP 1324, July 2010.

[56] G. Kutyniok, J. Lemvig, and W.-Q Lim. Compactly Supported Shearlets. Preprint 56, DFG-SPP 1324, July 2010.

[57] A. Zeiser. On the Optimality of the Inexact Inverse Iteration Coupled with Adaptive Finite Element Methods. Preprint 57, DFG-SPP 1324, July 2010.

[58] S. Jokar. Sparse Recovery and Kronecker Products. Preprint 58, DFG-SPP 1324, August 2010.

[59] T. Aboiyar, E. H. Georgoulis, and A. Iske. Adaptive ADER Methods Using Kernel-Based Polyharmonic Spline WENO Reconstruction. Preprint 59, DFG-SPP 1324, August 2010.

[60] O. G. Ernst, A. Mugler, H.-J. Starkloff, and E. Ullmann. On the Convergence of Generalized Polynomial Chaos Expansions. Preprint 60, DFG-SPP 1324, August 2010.

[61] S. Holtz, T. Rohwedder, and R. Schneider. On Manifolds of Tensors of Fixed TT-Rank. Preprint 61, DFG-SPP 1324, September 2010.

[62] J. Ballani, L. Grasedyck, and M. Kluge. Black Box Approximation of Tensors in Hierarchical Tucker Format. Preprint 62, DFG-SPP 1324, October 2010.

[63] M. Hansen. On Tensor Products of Quasi-Banach Spaces. Preprint 63, DFG-SPP 1324, October 2010.

[64] S. Dahlke, G. Steidl, and G. Teschke. Shearlet Coorbit Spaces: Compactly Supported Analyzing Shearlets, Traces and Embeddings. Preprint 64, DFG-SPP 1324, October 2010.

[65] W. Hackbusch. Tensorisation of Vectors and their Efficient Convolution. Preprint 65, DFG-SPP 1324, November 2010.

[66] P. A. Cioica, S. Dahlke, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Spatial Besov Regularity for Stochastic Partial Differential Equations on Lipschitz Domains. Preprint 66, DFG-SPP 1324, November 2010.

[67] E. Novak and H. Woźniakowski. On the Power of Function Values for the Approximation Problem in Various Settings. Preprint 67, DFG-SPP 1324, November 2010.

[68] A. Hinrichs, E. Novak, and H. Woźniakowski. The Curse of Dimensionality for Monotone and Convex Functions of Many Variables. Preprint 68, DFG-SPP 1324, November 2010.

[69] G. Kutyniok and W.-Q Lim. Image Separation Using Shearlets. Preprint 69, DFG-SPP 1324, November 2010.

[70] B. Jin and P. Maass. An Analysis of Electrical Impedance Tomography with Applications to Tikhonov Regularization. Preprint 70, DFG-SPP 1324, December 2010.

[71] S. Holtz, T. Rohwedder, and R. Schneider. The Alternating Linear Scheme for Tensor Optimisation in the TT Format. Preprint 71, DFG-SPP 1324, December 2010.

[72] T. Müller-Gronbach and K. Ritter. A Local Refinement Strategy for Constructive Quantization of Scalar SDEs. Preprint 72, DFG-SPP 1324, December 2010.

[73] T. Rohwedder and R. Schneider. An Analysis for the DIIS Acceleration Method used in Quantum Chemistry Calculations. Preprint 73, DFG-SPP 1324, December 2010.

[74] C. Bender and J. Steiner. Least-Squares Monte Carlo for Backward SDEs. Preprint 74, DFG-SPP 1324, December 2010.

[75] C. Bender. Primal and Dual Pricing of Multiple Exercise Options in Continuous Time. Preprint 75, DFG-SPP 1324, December 2010.

[76] H. Harbrecht, M. Peters, and R. Schneider. On the Low-rank Approximation by the Pivoted Cholesky Decomposition. Preprint 76, DFG-SPP 1324, December 2010.

[77] P. A. Cioica, S. Dahlke, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Adaptive Wavelet Methods for Elliptic Stochastic Partial Differential Equations. Preprint 77, DFG-SPP 1324, January 2011.

[78] G. Plonka, S. Tenorth, and A. Iske. Optimal Representation of Piecewise Hölder Smooth Bivariate Functions by the Easy Path Wavelet Transform. Preprint 78, DFG-SPP 1324, January 2011.

[79] A. Mugler and H.-J. Starkloff. On Elliptic Partial Differential Equations with Random Coefficients. Preprint 79, DFG-SPP 1324, January 2011.

[80] T. Müller-Gronbach, K. Ritter, and L. Yaroslavtseva. A Derandomization of the Euler Scheme for Scalar Stochastic Differential Equations. Preprint 80, DFG-SPP 1324, January 2011.

[81] W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive Petrov-Galerkin methods for first order transport equations. Preprint 81, DFG-SPP 1324, January 2011.

[82] K. Grella and C. Schwab. Sparse Tensor Spherical Harmonics Approximation in Radiative Transfer. Preprint 82, DFG-SPP 1324, January 2011.

[83] D.A. Lorenz, S. Schiffler, and D. Trede. Beyond Convergence Rates: Exact Inversion With Tikhonov Regularization With Sparsity Constraints. Preprint 83, DFG-SPP 1324, January 2011.

[84] S. Dereich, M. Scheutzow, and R. Schottstedt. Constructive quantization: Approximation by empirical measures. Preprint 84, DFG-SPP 1324, January 2011.

[85] S. Dahlke and W. Sickel. On Besov Regularity of Solutions to Nonlinear Elliptic Partial Differential Equations. Preprint 85, DFG-SPP 1324, January 2011.

[86] S. Dahlke, U. Friedrich, P. Maass, T. Raasch, and R.A. Ressel. An adaptive wavelet method for parameter identification problems in parabolic partial differential equations. Preprint 86, DFG-SPP 1324, January 2011.

[87] A. Cohen, W. Dahmen, and G. Welper. Adaptivity and Variational Stabilization for Convection-Diffusion Equations. Preprint 87, DFG-SPP 1324, January 2011.

[88] T. Jahnke. On Reduced Models for the Chemical Master Equation. Preprint 88, DFG-SPP 1324, January 2011.

[89] P. Binev, W. Dahmen, R. DeVore, P. Lamby, D. Savu, and R. Sharpley. Compressed Sensing and Electron Microscopy. Preprint 89, DFG-SPP 1324, March 2011.

[90] P. Binev, F. Blanco-Silva, D. Blom, W. Dahmen, P. Lamby, R. Sharpley, and T. Vogt. High Quality Image Formation by Nonlocal Means Applied to High-Angle Annular Dark Field Scanning Transmission Electron Microscopy (HAADF-STEM). Preprint 90, DFG-SPP 1324, March 2011.

[91] R. A. Ressel. A Parameter Identification Problem for a Nonlinear Parabolic Differential Equation. Preprint 91, DFG-SPP 1324, May 2011.

[92] G. Kutyniok. Data Separation by Sparse Representations. Preprint 92, DFG-SPP 1324, May 2011.

[93] M. A. Davenport, M. F. Duarte, Y. C. Eldar, and G. Kutyniok. Introduction to Compressed Sensing. Preprint 93, DFG-SPP 1324, May 2011.

[94] H.-C. Kreusler and H. Yserentant. The Mixed Regularity of Electronic Wave Functions in Fractional Order and Weighted Sobolev Spaces. Preprint 94, DFG-SPP 1324, June 2011.

[95] E. Ullmann, H. C. Elman, and O. G. Ernst. Efficient Iterative Solvers for Stochastic Galerkin Discretizations of Log-Transformed Random Diffusion Problems. Preprint 95, DFG-SPP 1324, June 2011.

[96] S. Kunis and I. Melzer. On the Butterfly Sparse Fourier Transform. Preprint 96, DFG-SPP 1324, June 2011.

[97] T. Rohwedder. The Continuous Coupled Cluster Formulation for the Electronic Schrödinger Equation. Preprint 97, DFG-SPP 1324, June 2011.

[98] T. Rohwedder and R. Schneider. Error Estimates for the Coupled Cluster Method. Preprint 98, DFG-SPP 1324, June 2011.

[99] P. A. Cioica and S. Dahlke. Spatial Besov Regularity for Semilinear Stochastic Partial Differential Equations on Bounded Lipschitz Domains. Preprint 99, DFG-SPP 1324, July 2011.

[100] L. Grasedyck and W. Hackbusch. An Introduction to Hierarchical (H-) Rank and TT-Rank of Tensors with Examples. Preprint 100, DFG-SPP 1324, August 2011.

[101] N. Chegini, S. Dahlke, U. Friedrich, and R. Stevenson. Piecewise Tensor Product Wavelet Bases by Extensions and Approximation Rates. Preprint 101, DFG-SPP 1324, September 2011.

[102] S. Dahlke, P. Oswald, and T. Raasch. A Note on Quarkonial Systems and Multilevel Partition of Unity Methods. Preprint 102, DFG-SPP 1324, September 2011.

[103] A. Uschmajew. Local Convergence of the Alternating Least Squares Algorithm For Canonical Tensor Approximation. Preprint 103, DFG-SPP 1324, September 2011.

[104] S. Kvaal. Multiconfigurational time-dependent Hartree method for describing particle loss due to absorbing boundary conditions. Preprint 104, DFG-SPP 1324, September 2011.

[105] M. Guillemard and A. Iske. On Groupoid C*-Algebras, Persistent Homology and Time-Frequency Analysis. Preprint 105, DFG-SPP 1324, September 2011.

[106] A. Hinrichs, E. Novak, and H. Woźniakowski. Discontinuous information in the worst case and randomized settings. Preprint 106, DFG-SPP 1324, September 2011.

[107] M. Espig, W. Hackbusch, A. Litvinenko, H. Matthies, and E. Zander. Efficient Analysis of High Dimensional Data in Tensor Formats. Preprint 107, DFG-SPP 1324, September 2011.

[108] M. Espig, W. Hackbusch, S. Handschuh, and R. Schneider. Optimization Problems in Contracted Tensor Networks. Preprint 108, DFG-SPP 1324, October 2011.

[109] S. Dereich, T. Müller-Gronbach, and K. Ritter. On the Complexity of Computing Quadrature Formulas for SDEs. Preprint 109, DFG-SPP 1324, October 2011.

[110] D. Belomestny. Solving optimal stopping problems by empirical dual optimization and penalization. Preprint 110, DFG-SPP 1324, November 2011.

[111] D. Belomestny and J. Schoenmakers. Multilevel dual approach for pricing American style derivatives. Preprint 111, DFG-SPP 1324, November 2011.

[112] T. Rohwedder and A. Uschmajew. Local convergence of alternating schemes for optimization of convex problems in the TT format. Preprint 112, DFG-SPP 1324, December 2011.

[113] T. Görner, R. Hielscher, and S. Kunis. Efficient and accurate computation of spherical mean values at scattered center points. Preprint 113, DFG-SPP 1324, December 2011.

[114] Y. Dong, T. Görner, and S. Kunis. An iterative reconstruction scheme for photoacoustic imaging. Preprint 114, DFG-SPP 1324, December 2011.

[115] L. Kämmerer. Reconstructing hyperbolic cross trigonometric polynomials by sampling along generated sets. Preprint 115, DFG-SPP 1324, February 2012.

[116] H. Chen and R. Schneider. Numerical analysis of augmented plane waves methods for full-potential electronic structure calculations. Preprint 116, DFG-SPP 1324, February 2012.

[117] J. Ma, G. Plonka, and M.Y. Hussaini. Compressive Video Sampling with Approximate Message Passing Decoding. Preprint 117, DFG-SPP 1324, February 2012.

[118] D. Heinen and G. Plonka. Wavelet shrinkage on paths for scattered data denoising. Preprint 118, DFG-SPP 1324, February 2012.

[119] T. Jahnke and M. Kreim. Error bound for piecewise deterministic processes modeling stochastic reaction systems. Preprint 119, DFG-SPP 1324, March 2012.

[120] C. Bender and J. Steiner. A-posteriori estimates for backward SDEs. Preprint 120, DFG-SPP 1324, April 2012.

[121] M. Espig, W. Hackbusch, A. Litvinenkoy, H.G. Matthiesy, and P. Wähnert. Efficient low-rank approximation of the stochastic Galerkin matrix in tensor formats. Preprint 121, DFG-SPP 1324, May 2012.

[122] O. Bokanowski, J. Garcke, M. Griebel, and I. Klompmaker. An adaptive sparse grid semi-Lagrangian scheme for first order Hamilton-Jacobi Bellman equations. Preprint 122, DFG-SPP 1324, June 2012.

[123] A. Mugler and H.-J. Starkloff. On the convergence of the stochastic Galerkin method for random elliptic partial differential equations. Preprint 123, DFG-SPP 1324, June 2012.

[124] P.A. Cioica, S. Dahlke, N. Döhring, U. Friedrich, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R.L. Schilling. On the convergence analysis of Rothe's method. Preprint 124, DFG-SPP 1324, July 2012.

[125] P. Binev, A. Cohen, W. Dahmen, and R. DeVore. Classification Algorithms using Adaptive Partitioning. Preprint 125, DFG-SPP 1324, July 2012.

[126] C. Lubich, T. Rohwedder, R. Schneider, and B. Vandereycken. Dynamical approximation of hierarchical Tucker and Tensor-Train tensors. Preprint 126, DFG-SPP 1324, July 2012.

[127] M. Kovács, S. Larsson, and K. Urban. On Wavelet-Galerkin methods for semilinear parabolic equations with additive noise. Preprint 127, DFG-SPP 1324, August 2012.

[128] M. Bachmayr, H. Chen, and R. Schneider. Numerical analysis of Gaussian approximations in quantum chemistry. Preprint 128, DFG-SPP 1324, August 2012.

[129] D. Rudolf. Explicit error bounds for Markov chain Monte Carlo. Preprint 129, DFG-SPP 1324, August 2012.

[130] P.A. Cioica, K.-H. Kim, K. Lee, and F. Lindner. On the $L_q(L_p)$-regularity and Besov smoothness of stochastic parabolic equations on bounded Lipschitz domains. Preprint 130, DFG-SPP 1324, December 2012.

[131] M. Hansen. $n-$term Approximation Rates and Besov Regularity for Elliptic PDEs on Polyhedral Domains. Preprint 131, DFG-SPP 1324, December 2012.

[132] R. E. Bank and H. Yserentant. On the H^1 -stability of the L_2 -projection onto finite element spaces. Preprint 132, DFG-SPP 1324, December 2012.

[133] M. Gnewuch, S. Mayer, and K. Ritter. On Weighted Hilbert Spaces and Integration of Functions of Infinitely Many Variables. Preprint 133, DFG-SPP 1324, December 2012.

[134] D. Crisan, J. Diehl, P.K. Friz, and H. Oberhauser. Robust Filtering: Correlated Noise and Multidimensional Observation. Preprint 134, DFG-SPP 1324, January 2013.

[135] Wolfgang Dahmen, Christian Plesken, and Gerrit Welper. Double Greedy Algorithms: Reduced Basis Methods for Transport Dominated Problems. Preprint 135, DFG-SPP 1324, February 2013.

[136] Aicke Hinrichs, Erich Novak, Mario Ullrich, and Henryk Wozniakowski. The Curse of Dimensionality for Numerical Integration of Smooth Functions. Preprint 136, DFG-SPP 1324, February 2013.

[137] Markus Bachmayr, Wolfgang Dahmen, Ronald DeVore, and Lars Grasedyck. Approximation of High-Dimensional Rank One Tensors. Preprint 137, DFG-SPP 1324, March 2013.

[138] Markus Bachmayr and Wolfgang Dahmen. Adaptive Near-Optimal Rank Tensor Approximation for High-Dimensional Operator Equations. Preprint 138, DFG-SPP 1324, April 2013.

[139] Felix Lindner. Singular Behavior of the Solution to the Stochastic Heat Equation on a Polygonal Domain. Preprint 139, DFG-SPP 1324, May 2013.

[140] Stephan Dahlke, Dominik Lellek, Shiu Hong Lui, and Rob Stevenson. Adaptive Wavelet Schwarz Methods for the Navier-Stokes Equation. Preprint 140, DFG-SPP 1324, May 2013.

[141] Jonas Ballani and Lars Grasedyck. Tree Adaptive Approximation in the Hierarchical Tensor Format. Preprint 141, DFG-SPP 1324, June 2013.

[142] Harry Yserentant. A short theory of the Rayleigh-Ritz method. Preprint 142, DFG-SPP 1324, July 2013.

[143] M. Hefter and K. Ritter. On Embeddings of Weighted Tensor Product Hilbert Spaces. Preprint 143, DFG-SPP 1324, August 2013.

[144] M. Altmayer and A. Neuenkirch. Multilevel Monte Carlo Quadrature of Discontinuous Payoffs in the Generalized Heston Model using Malliavin Integration by Parts. Preprint 144, DFG-SPP 1324, August 2013.

[145] L. Kämmerer, D. Potts, and T. Volkmer. Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling. Preprint 145, DFG-SPP 1324, September 2013.

[146] C. Bender, N. Schweizer, and J. Zhuo. A primal-dual algorithm for BSDEs. Preprint 146, DFG-SPP 1324, October 2013.

[147] D. Rudolf. Hit-and-run for numerical integration. Preprint 147, DFG-SPP 1324, October 2013.

[148] D. Rudolf and M. Ullrich. Positivity of hit-and-run and related algorithms. Preprint 148, DFG-SPP 1324, October 2013.

[149] L. Grasedyck, M. Kluge, and S. Krämer. Alternating Directions Fitting (ADF) of Hierarchical Low Rank Tensors. Preprint 149, DFG-SPP 1324, October 2013.

[150] F. Filbir, S. Kunis, and R. Seyfried. Effective discretization of direct reconstruction schemes for photoacoustic imaging in spherical geometries. Preprint 150, DFG-SPP 1324, November 2013.

[151] E. Novak, M. Ullrich, and H. Woźniakowski. Complexity of Oscillatory Integration for Univariate Sobolev Spaces. Preprint 151, DFG-SPP 1324, November 2013.