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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Pathwise Dynamic Programming

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Christian Bender[1], Christian Gärtner[1], and Nikolaus Schweizer[2]

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Abstract

We present a novel method for deriving tight Monte Carlo confidence intervals for solutions of stochastic dynamic programming equations. Taking some approximate solution to the equation as an input, we construct pathwise recursions with a known bias. Suitably coupling the recursions for lower and upper bounds ensures that the method is applicable even when the dynamic program does not satisfy a comparison principle. We apply our method to two nonlinear option pricing problems, pricing under bilateral counterparty risk and pricing under uncertain volatility.

Keywords: stochastic dynamic programming; Monte Carlo; confidence bounds; option pricing

1 Introduction

We study the problem of computing Monte Carlo confidence intervals for solutions of discrete time, finite horizon stochastic dynamic programming equations. There is a random terminal value, and a nonlinear recursion which allows to compute the value at a given time from expectations about the value one step ahead. Equations of this type are highly prominent in the analysis of multistage sequential decision problems under uncertainty (Bertsekas, 2005; Powell, 2011). Yet they also arise in a variety of further applications such as financial option pricing (e.g. Guyon and Henry-Labordère, 2013), the evaluation of recursive utility functionals (e.g. Kraft and Seifried, 2014) or the numerical solution of partial differential equations (e.g. Fahim et al., 2011).

The key challenge when computing solutions to stochastic dynamic programs numerically stems from a high order nesting of conditional expectations operators in the backward recursion. The solution at each time step depends on an expectation of what happens one step ahead, which in turn depends on expectations of what happens at later dates. If the system is driven by a Markov process with a high-dimensional state space – as is the case in typical applications – a naive numerical approach quickly runs into the curse of dimensionality. In practice, conditional expectations thus need to be replaced by an approximate conditional expectations operator which can be nested several times at moderate computational costs, e.g., by a least-squares Monte Carlo approximation. For an introduction and overview of this “approximate dynamic programming” approach, see, e.g., Powell (2011). The error induced by these approximations is typically hard to quantify and control. This motivates us to develop a posteriori criteria which use an approximate

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or heuristic solution to the dynamic program as an input in order to compute tight upper and lower bounds on the true solution. In this context, "tight" means that the bounds match when the input approximation coincides with the true solution.

Specifically, we study stochastic dynamic programming equations of the form

$$Y_j = G_j(E_j[\beta_{j+1}Y_{j+1}], F_j(E_j[\beta_{j+1}Y_{j+1}])) \tag{1}$$

with random terminal condition $Y_J = \xi$. Here, $E_j[\cdot]$ denotes the conditional expectation given the information at time step j. F_j and G_j are real-valued, adapted, random functions. F_j is convex while G_j is concave and increasing in its second argument. In particular, a possible dependence on state and decision variables is modeled implicitly through the functions F_j and G_j. The stochastic weight vectors β_j form an adapted process, i.e., unlike F_j and G_j, β_{j+1} is known at time $j+1$ but not yet at time j. Such weight processes frequently arise in dynamic programs for financial option valuation problems where, roughly speaking, they are associated with the first two derivatives, Delta and Gamma, of the value process Y. Similarly, Fahim et al. (2011) derive time discretization schemes of the form (1) for fully nonlinear second-order parabolic partial differential equations which involve stochastic weights in the approximation of derivatives.

There are two main motivations for assuming the concave-convex structure of the right hand side of (1). First, since convexity and concavity may stem, respectively, from maximization or minimization, the equation is general enough to cover not only dynamic programs associated with classical optimization problems but also dynamic programs arising from certain stochastic two-player games or robust optimization problems under model uncertainty. Second, many functions which are themselves neither concave nor convex may be written in this concave-convex form. This can be seen, for instance, in the application to bilateral counterparty credit risk below.

Traditionally, a posteriori criteria for this type of recursion were not derived directly from equation (1) but rather from a primal-dual pair of optimization problems associated with it. For instance, with the choices $G_j(z, y) = y$, $F_j(z) = \max(S_j, z)$, $\beta_j = 1$ and $\xi = S_J$, (1) becomes the dynamic programming equation associated with the optimal stopping problem for a process $(S_j)_j$. From an approximate solution of (1) one can conclude an approximation of the optimal stopping strategy. Testing this strategy by simulation yields lower bounds on the value of the optimal stopping problem. The derivation of dual upper bounds starts by considering strategies which allow to look into the future, i.e., by considering strategies which allow to optimize pathwise rather than in conditional expectation. The best strategy of this type is easy to simulate and implies an upper bound on the value of the usual optimal stopping problem. This bound can be made tight by penalizing the use of future information in an appropriate way. Approximately optimal information penalties can be derived from approximate solutions to (1). Combining the resulting low-biased and high-biased Monte Carlo estimators for Y_0 yields Monte Carlo confidence intervals for the solution of the dynamic program. This information relaxation approach was developed in the context of optimal stopping independently by Rogers (2002) and Haugh and Kogan (2004). The approach was extended to general stochastic dynamic optimization problems by Brown et al. (2010) and Rogers (2007).

Recently, Bender et al. (2015) have proposed a posteriori criteria which are derived directly from a dynamic programming recursion like (1). The obvious advantage is that, in principle, this approach requires neither knowledge nor existence of associated primal and dual optimization problems. This paper generalizes the approach of Bender et al. (2015) in various directions. For example, we merely require that the functions F_j and G_j are of polynomial growth while the corresponding condition in the latter paper is Lipschitz continuity. Moreover, we assume that the weights β_j are sufficiently integrable rather than bounded, and introduce the concave-convex functional form of the right hand side of (1) which is more flexible than the functional forms considered there.

Conceptually, our main departure from the approach of Bender et al. (2015) is that we do not require that the recursion (1) satisfies a comparison principle. Suppose that two adapted processes Y^{low} and Y^{up} fulfill the analogs of (1) with "=" replaced, respectively, by "$\leq$" and "$\geq$". We call such processes subsolutions and supersolutions to (1). The comparison principle postulates that subsolutions are always smaller than supersolutions. Relying on such a comparison principle, the approach of Bender et al. (2015) mimics the classical situation in the sense that their lower and upper bounds can be interpreted as stemming, respectively, from a primal and a dual optimization problem constructed from (1).

The bounds of the present paper apply regardless of whether a comparison principle holds or not. This increased applicability is achieved at negligible additional numerical costs. The key idea is to construct a pathwise recursion associated with particular pairs of super- and subsolutions. These pairs remain ordered even when such an ordering is not a generic property of super- and subsolutions, i.e., when comparison is violated in general. Roughly speaking, whenever a violation of comparison threatens to reverse the order of the upper and lower bound, the upper and lower bound are simply exchanged on the respective path. Consequently, lower and upper bounds must always be computed simultaneously. In particular, they can no longer be viewed as the respective solutions of distinct primal and dual optimization problems.

For some applications like optimal stopping, the comparison principle is not an issue. For others like the nonlinear pricing applications studied in Bender et al. (2015), the principle can be set in force by a relatively mild truncation of the stochastic weights β. Dispensing with the comparison principle allows us to avoid any truncation and the corresponding truncation error. This matters because there is also a class of applications where the required truncation levels are so low that truncation would fundamentally alter the problem. This concerns, in particular, backward recursions associated with fully nonlinear second-order partial differential equations such as the problem of pricing under volatility uncertainty which serves as a running example throughout the paper. Solving equation (1) for the uncertain volatility model is well-known as a challenging numerical problem (Guyon and Henry-Labordère, 2011; Alanko and Avellaneda, 2013), making an a posteriori evaluation of solutions particularly desirable.

The paper is organized as follows: Section 2 introduces the setting and our key assumptions. Section 3 presents our main results. We first explain a construction of a pair of upper and lower bounds which relies on the comparison principle and discuss the restrictiveness of the principle. Afterwards we state the key result of the paper, a new pair of upper and lower bounds which is applicable in the absence of a comparison principle. In Section 4, we relate our results to the information relaxation duals of Brown et al. (2010). In particular, we show that our bounds in the presence of the comparison principle can be reinterpreted in terms of information relaxation bounds for stochastic two-player zero-sum games as recently studied in Haugh and Wang (2015). Finally, in Section 5, we apply our results in the context of two nonlinear valuation problems, option pricing in a four-dimensional interest rate model with bilateral counterparty risk and in the uncertain volatility models. While such nonlinearities in pricing due to counterparty risk and model uncertainty have received increased attention since the financial crisis, Monte Carlo confidence bounds for these two problems were previously unavailable. All proofs are postponed to the Appendix.

2 Preliminaries

Throughout the paper we study the following type of concave-convex dynamic programming equation on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\dots J}, P)$ in discrete time:

$$
\begin{aligned}
Y_J &= \xi, \\
Y_j &= G_j(E_j[\beta_{j+1}Y_{j+1}], F_j(E_j[\beta_{j+1}Y_{j+1}])), \quad j = J-1, \dots, 0
\end{aligned}
\tag{2}
$$

where $E_j[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_j$. We assume

(C): For every $j = 0, \dots, J-1$, $G_j : \Omega \times \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$ and $F_j : \Omega \times \mathbb{R}^D \to \mathbb{R}$ are measurable and, for every $(z, y) \in \mathbb{R}^D \times \mathbb{R}$, the processes $(j, \omega) \mapsto G_j(\omega, z, y)$ and $(j, \omega) \mapsto F_j(\omega, z)$ are adapted. Moreover, for every $j = 0, \dots, J-1$ and $\omega \in \Omega$, the map $(z, y) \mapsto G_j(\omega, z, y)$ is concave in (z, y) and non-decreasing in y, and the map $z \mapsto F_j(\omega, z)$ is convex in z.

(R):

- G and F are of polynomial growth in (z, y) in the following sense: There exist a constant $q \geq 0$ and a nonnegative adapted process (α_j) such that for all $(z, y) \in \mathbb{R}^{D+1}$ and $j = 0, \dots, J-1$

$$
|G_j(z, y)| + |F_j(z)| \leq \alpha_j(1 + |z|^q + |y|^q), \quad P\text{-a.s.},
$$

  and $\alpha_j \in L^p(\Omega, P)$ for every $p \geq 1$.
- $\beta = (\beta_j)_{j=1,\dots,J}$ is an adapted process such that $\beta_j \in L^p(\Omega, P)$ for every $p \geq 1$ and $j = 1, \dots, J$.
- The terminal condition ξ is an $\mathcal{F}_J$-measurable random variable such that $\xi \in L^p(\Omega, P)$ for every $p \geq 1$.

In the following, we denote by

- $L^{\infty-}(\mathbb{R}^N)$ the set of $\mathbb{R}^N$-valued random variables that are in $L^p(\Omega, P)$ for all $p \geq 1$.
- $L^{\infty-}_j(\mathbb{R}^N)$ the set of $\mathcal{F}_j$-measurable random variables that are in $L^{\infty-}(\mathbb{R}^N)$.
- $L^{\infty-}_{ad}(\mathbb{R}^N)$ the set of adapted processes Z such that $Z_j \in L^{\infty-}_j(\mathbb{R}^N)$ for every $j = 0, \dots, J$.

Thanks to the integrability assumptions on the terminal condition ξ and the weight process β and thanks to the polynomial growth condition on F and G, it is straightforward to check recursively that the (P-a.s. unique) solution Y to (2) belongs to $L^{\infty-}_{ad}(\mathbb{R})$ under assumptions (C) and (R).

The aim of the present paper is to devise a pathwise dynamic programming approach, which avoids the use of nested conditional expectations, in order to construct tight upper and lower bounds on the solution Y_j. Extending the ideas in Bender et al. (2015), this construction will be based on the concept of supersolutions and subsolutions to the dynamic program (2).

Definition 2.1. A process Y^{up} (resp. Y^{low}) $\in L^{\infty-}_{ad}(\mathbb{R})$ is called *supersolution* (resp. *subsolution*) to the dynamic program (2) if $Y^{up}_J \geq Y_J$ (resp. $Y^{low}_J \leq Y_J$) and for every $j = 0, \dots, J-1$ it holds

$$
Y^{up}_j \geq G_j([E_j[\beta_{j+1}Y^{up}_{j+1}], F_j(E_j[\beta_{j+1}Y^{up}_{j+1}])), \quad P\text{-a.s.}
$$

(and with ' $\geq$' replaced by ' $\leq$' for a subsolution).

Before we explain a generic way to construct super- and subsolutions to the concave-convex dynamic program (2) in Section 3, we first need to fix some more notation. $(\cdot)_{\pm}$ denotes, respectively, the positive part and the negative part of a real-valued function. $(\cdot)^{\top}$ denotes matrix transposition.

The convex conjugate of F_j is given by

$$F_j^{\#}(u) := \sup_{z \in \mathbb{R}^D} (u^{\top} z - F_j(z)), \quad u \in \mathbb{R}^D.$$

Note that $F_j^{\#}$ can take the value $+\infty$ and that the maximization takes place pathwise, i.e., ω by ω. Analogously, for G_j the concave conjugate can be defined in terms of the convex conjugate of $-G_j$ as

$$\begin{aligned} G_j^{\#}\left(v^{(1)}, v^{(0)}\right) &:= -(-G_j)^{\#}\left(-v^{(1)}, -v^{(0)}\right) \\ &= \inf_{(z,y) \in \mathbb{R}^{D+1}} \left(\left(v^{(1)}\right)^{\top} z + v^{(0)} y - G_j(z,y)\right), \quad \left(v^{(1)}, v^{(0)}\right) \in \mathbb{R}^{D+1}, \end{aligned}$$

which can take the value $-\infty$.

We next introduce sets of adapted 'controls' in terms of F and G by

$$\begin{aligned} \mathcal{A}_j^F &= \left\{ (r_i)_{i=j,\ldots,J-1} \middle| r_i \in L_i^{\infty-}(\mathbb{R}^D),\ F_i^{\#}(r_i) \in L^{\infty-}(\mathbb{R}) \text{ for } i = j, \ldots, J-1 \right\}, \\ \mathcal{A}_j^G &= \left\{ \left(\rho_i^{(1)}, \rho_i^{(0)}\right)_{i=j,\ldots,J-1} \middle| \left(\rho_i^{(1)}, \rho_i^{(0)}\right) \in L_i^{\infty-}(\mathbb{R}^{D+1}), \right. \\ &\qquad \left. G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \in L^{\infty-}(\mathbb{R}) \text{ for } i = j, \ldots, J-1 \right\}. \end{aligned} \tag{3}$$

Finally, we denote by $\mathcal{M}_D$ the set of martingales in $L_{ad}^{\infty-}(\mathbb{R}^D)$.

We note that, by continuity of F_i,

$$F_i^{\#}(r_i) = \sup_{z \in \mathbb{Q}^D} (r_i^{\top} z - F_i(z))$$

is $\mathcal{F}_i$-measurable for every $r \in \mathcal{A}_j^F$ and $i = j, \ldots, J-1$ — and so is $G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)})$ for $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$. Moreover, the integrability condition on the controls requires that $F_i^{\#}(r_i) < \infty$ and $G_i^{\#}(\rho_i^{(1)}, \rho_i^{(0)}) > -\infty$ P-almost surely. In particular, by the monotonicity assumption on G in the y-variable, we observe that $\rho_i^{(0)} \geq 0$ P-almost surely.

As a direct consequence of the following lemma, we observe that the sets of controls introduced above are nonempty under the standing assumptions (C) and (R).

Lemma 2.2. *Fix $j \in \{0, \ldots, J-1\}$ and let $f_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a mapping such that, for every $\omega \in \Omega$, the map $x \mapsto f_j(\omega, x)$ is convex, and for every $x \in \mathbb{R}^N$, the map $\omega \mapsto f_j(\omega, x)$ is $\mathcal{F}_j$-measurable. Moreover, suppose that f_j satisfies the following polynomial growth condition: There are a constant $q \geq 0$ and a nonnegative random variable $\alpha_j \in L_j^{\infty-}(\mathbb{R})$ such that*

$$|f_j(x)| \leq \alpha_j(1 + |x|^q), \quad P\text{-a.s.},$$

for every $x \in \mathbb{R}^N$. Then, for every $\overline{Z} \in L_j^{\infty-}(\mathbb{R}^N)$ there exists a random variable $\overline{\rho}_j \in L_j^{\infty-}(\mathbb{R}^N)$ such that $f_j^{\#}\left(\overline{\rho}_j\right) \in L_j^{\infty-}(\mathbb{R})$ and

$$f_j(\overline{Z}) = \overline{\rho}_j^{\top} \overline{Z} - f_j^{\#}(\overline{\rho}_j), \quad P\text{-a.s.} \tag{4}$$

In order to simplify the exposition, we shall use the following convention for the remainder of the paper: Unless otherwise noted, all equations and inequalities are supposed to hold P-almost surely.

3 Main results

We begin this section with an example of a generic construction of super- and subsolutions in terms of admissible controls and martingales. This construction is similar in spirit but more general than the one in Bender et al. (2015). We call it an example to emphasize that its main purpose is to motivate and prepare the modified construction introduced later on. In particular, we show in Theorems 3.2 and 3.3 that the super- and subsolutions of the example only imply upper and lower bounds on the solution of the dynamic program under rather restrictive conditions, namely, the assumption of a comparison principle. This observation motivates the main result of this section: In Theorem 3.5, we propose an alternative pathwise recursion which is associated with a pair of super- and subsolutions and yields valid upper and lower bounds, even when the dynamic programming equation violates the general comparison principle.

Example 3.1. (i) For a given $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $M \in \mathcal{M}_D$ define $\Theta_j^{up} = \Theta_j^{up}(\rho^{(1)}, \rho^{(0)}, M)$ by the pathwise dynamic program

$$\Theta_j^{up} = \left(\rho_j^{(1)}\right)^\top \left(\beta_{j+1}\Theta_{j+1}^{up} - (M_{j+1} - M_j)\right) + \rho_j^{(0)} F_j(\beta_{j+1}\Theta_{j+1}^{up} - (M_{j+1} - M_j)) - G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}),$$

for $j = J-1, \ldots, 0$, initiated at $\Theta_J^{up} = \xi$. Then, $Y_j^{up} := E_j[\Theta_j^{up}]$ is a supersolution to the concave-convex dynamic program (2). Indeed, we observe recursively that $\Theta_j^{up} \in L^{\infty-}(\mathbb{R})$ for every $j = J, \ldots, 0$. Moreover, by the tower property of the conditional expectation, Jensen's inequality, the convexity of F, and the martingale property of M, we have

$$Y_j^{up} \geq \left(\rho_j^{(1)}\right)^\top E_j[\beta_{j+1}Y_{j+1}^{up}] + \rho_j^{(0)} F_j(E_j[\beta_{j+1}Y_{j+1}^{up}]) - G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}).$$

Here, we also used that $(\rho_j^{(1)}, \rho_j^{(0)})$ is $\mathcal{F}_j$-measurable and that $\rho_j^{(0)} \geq 0$ by the monotonicity of G in the y-variable. We next exploit that $G_j^{\#\#} = G_j$, see, e.g., Theorem 12.2 in Rockafellar (1970), because G is for fixed (j, ω) concave and closed in $(v^{(1)}, v^{(0)})$. Hence, for every $j = 0, \ldots, J-1$, $(z, y) \in \mathbb{R}^{D+1}$, and every $\omega \in \Omega$,

$$G_j(\omega, z, y) = \inf_{(v^{(1)}, v^{(0)}) \in \mathbb{R}^{D+1}} \left(\left(v^{(1)}\right)^\top z + v^{(0)} y - G_j^{\#}(\omega, v^{(1)}, v^{(0)}) \right), \tag{5}$$

which, in particular, implies

$$Y_j^{up} \geq G_j(E_j[\beta_{j+1}Y_{j+1}^{up}], F_j(E_j[\beta_{j+1}Y_{j+1}^{up}])).$$

(ii) For a given $r \in \mathcal{A}_0^F$ and $M \in \mathcal{M}_D$ define $\Theta_j^{low} = \Theta_j^{low}(r, M)$ for $j = J-1, \ldots, 0$ by the pathwise dynamic program

$$\Theta_j^{low} = G_j\left(\beta_{j+1}\Theta_{j+1}^{low} - (M_{j+1} - M_j), r_j^\top \left(\beta_{j+1}\Theta_{j+1}^{low} - (M_{j+1} - M_j)\right) - F_j^{\#}(r_j)\right)$$

initiated at $\Theta_J^{low} = \xi$. Then, $Y_j^{low} := E_j[\Theta_j^{low}]$ is a subsolution of (2), because by concavity of G and Jensen's inequality (similarly as in (i))

$$Y_j^{low} \leq G_j\left(E_j[\beta_{j+1}Y_{j+1}^{low}], r_j^\top E_j[\beta_{j+1}Y_{j+1}^{low}] - F_j^{\#}(r_j)\right).$$

We finally observe, by convexity and closedness of F_j, that for every $j = 0, \ldots, J-1$, $z \in \mathbb{R}^D$, and every $\omega \in \Omega$

$$F_j(\omega, z) = \sup_{u \in \mathbb{R}^D} \left(u^\top z - F_j^{\#}(\omega, u)\right), \tag{6}$$

which in view of the monotonicity of G in the y-variable now implies

$$Y_j^{low} \leq G_j\left(E_j[\beta_{j+1}Y_{j+1}^{low}], F_j(E_j[\beta_{j+1}Y_{j+1}^{low}])\right).$$

The key advantage of the super- and subsolution constructed in the example is that, once the controls and the martingale are chosen, the recursion formulas can be computed pathwise and only one conditional expectation must be calculated (numerically) at the end of the recursion, instead of approximating nested conditional expectations in each recursion step, which is required in the original dynamic program (2).

In order to derive upper and lower bounds on the solution of the original dynamic program based on the pathwise dynamic programs in Example 3.1 we shall assume, for the moment, the following *comparison principle*:

(Comp): For every supersolution Y^{up} and every subsolution Y^{low} to the dynamic program (2) it holds that

$$Y_j^{up} \geq Y_j^{low}, \quad P\text{-a.s.}, \quad \text{for every } j = 0, \ldots, J.$$

Under this additional assumption we have the following result.

Theorem 3.2. *Suppose (C), (R), and (Comp). Then, for every $j = 0, \ldots, J$,*

$$Y_j = \operatorname*{essinf}_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G,\ M \in \mathcal{M}_D} E_j[\Theta_j^{up}(\rho^{(1)}, \rho^{(0)}, M)] = \operatorname*{esssup}_{r \in \mathcal{A}_j^F,\ M \in \mathcal{M}_D} E_j[\Theta_j^{low}(r, M)],\ P\text{-a.s.},$$

where Θ^{up} and Θ^{low} are the pathwise dynamic programs defined in Example 3.1. Moreover,

$$Y_j = \Theta_j^{up}(\rho^{(1,*)}, \rho^{(0,*)}, M^*) = \Theta_j^{low}(r^*, M^*) \tag{7}$$

P-almost surely, for every $(\rho^{(1,*)}, \rho^{(0,*)}) \in \mathcal{A}_j^G$ and $r^* \in \mathcal{A}_j^F$ satisfying the duality relations

$$\begin{aligned}\left(\rho_i^{(1,*)}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] + \rho_i^{(0,*)} F_i\left(E_i[\beta_{i+1}Y_{i+1}]\right) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right) \\ = G_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right], F_i(E_i[\beta_{i+1}Y_{i+1}])\right)\end{aligned} \tag{8}$$

and

$$\left(r_i^*\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] - F_i^{\#}\left(r_i^*\right) = F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right) \tag{9}$$

P-almost surely for every $i = j, \ldots, J-1$, and with M^* being the Doob martingale of βY, i.e.,

$$M_k^* = \sum_{i=0}^{k-1} \beta_{i+1}Y_{i+1} - E_i[\beta_{i+1}Y_{i+1}], \quad P\text{-a.s., for every } k = 0, \ldots J.$$

Concerning the optimizers, we emphasize that, by Lemma 2.2, there are admissible controls $(\rho^{(1,*)}, \rho^{(0,*)}) \in \mathcal{A}_j^G$ and $r^* \in \mathcal{A}_j^F$ which fulfill the duality relations (8) and (9) and, that M^* obviously belongs to $\mathcal{M}_D$. For the *proof* of Theorem 3.2 we note that by Example 3.1 and the assumed comparison principle, we obtain

$$E_j[\Theta_j^{low}(r, M)] \leq Y_j \leq E_j[\Theta_j^{up}(\rho^{(1)}, \rho^{(0)}, M)]$$

for every $M \in \mathcal{M}_D$, $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$, and $r \in \mathcal{A}_j^F$. So it only remains to prove (7), which follows by a rather straightforward backward induction, which we spell out in a more general context in Proposition A.2 below.

The duality relations (8) and (9) provide some guidance on how to choose nearly optimal controls in numerical implementations of the method: When an approximate solution $\tilde{Y}$ of the dynamic program is available, controls can be chosen such that they (approximately) fulfill the analogs of (8) and (9) with $\tilde{Y}$ in place of the unknown true solution. Likewise, M can be chosen as an approximate Doob martingale of $\beta\tilde{Y}$.

Evidently, the assertion of Theorem 3.2 crucially hinges on the assumption of the comparison principle. We next provide equivalent conditions for this principle in the special case of a convex dynamic program, i.e., when $G_i(z, y) = y$. This characterization shows that the comparison principle can be rather restrictive and is our motivation to remove it below through a more careful construction of suitable pathwise dynamic programs.

Theorem 3.3. *Suppose (R) and that the dynamic program is convex, i.e., (C) is in force with* $G_i(z, y) = y$ *for* $i = 0, \ldots, J-1$. *Then, the following assertions are equivalent:*
(a) The comparison principle (Comp) holds.
(b) For every $r \in \mathcal{A}_0^F$ *the following positivity condition is fulfilled: For every* $i = 0, \ldots, J-1$

$$r_i^\top \beta_{i+1} \geq 0, \quad P\text{-a.s.}$$

(c) For every $j = 0, \ldots, J-1$ *and any two random variables* $Y^{(1)}, Y^{(2)} \in L^{\infty-}(\mathbb{R})$ *with* $Y^{(1)} \geq Y^{(2)}$ P-a.s., *the monotonicity condition*

$$F_j(E_j[\beta_{j+1}Y^{(1)}]) \geq F_j(E_j[\beta_{j+1}Y^{(2)}]), \quad P\text{-a.s.},$$

is satisfied.

Theorem 3.3 provides three equivalent formulations for the comparison principle. Formulation (b) illustrates the restrictiveness of the principle most clearly. For instance, when β contains unbounded entries with random sign, comparison can only hold in degenerate cases. In some applications, problems of this type can be resolved by truncation of the weights at the cost of a small additional error. Yet in applications such as pricing under volatility uncertainty, the comparison principle may fail (even under truncation):

Example 3.4. The pricing problem of an option with payoff ξ at maturity T under uncertain volatility leads, after a time discretization, to a dynamic program of the form

$$Y_j = \max_{\rho_j \in \left\{\frac{1}{2}\left(\frac{\sigma_{low}^2}{\hat{\rho}^2}-1\right), \frac{1}{2}\left(\frac{\sigma_{up}^2}{\hat{\rho}^2}-1\right)\right\}} E_j\left[\left(1 + \rho_j\left(\frac{\Delta W_{j+1}^2}{\Delta_{j+1}} - \hat{\rho}\Delta W_{j+1} - 1\right)\right)Y_{j+1}\right], \quad Y_J = \xi,$$

where $0 = t_0 < t_1 < \cdots < t_J = T$ is a partition of $[0, T]$, $\Delta_j = t_j - t_{j-1}$, $\Delta W_j = W_{t_j} - W_{t_{j-1}}$ are increments of a Brownian motion W, and E_j denotes the expectation conditional on the information generated by the Brownian motion up to time t_j. The uncertain volatility may vary within the interval $[\sigma_{low}, \sigma_{up}]$, and $\hat{\rho}$ is a reference volatility of one's choice (under which the sample paths are generated in the numerical approximation later on). For more details we refer to Guyon and Henry-Labordère (2011) and Section 5.2 below. In this example, we can choose

$$\beta_{j+1} = \left(1, \frac{\Delta W_{j+1}^2}{\Delta_{j+1}^2} - \hat{\rho}\frac{\Delta W_{j+1}}{\Delta_{j+1}} - \frac{1}{\Delta_{j+1}}\right)^\top.$$

We then observe thanks to condition (c) in Theorem 3.3, that comparison boils down to the requirement that the prefactor of Y_{j+1} in the equation for Y_j is P-almost surely nonnegative for both of the feasible values of ρ_j. For $\rho_j > 1$, this requirement is violated for realizations of ΔW_j sufficiently close to zero, while for $\rho_j < 0$ violations occur for sufficiently negative realizations of the Brownian increment – and this violation also takes place if one truncates the Brownian increments at $\pm const.\sqrt{\Delta_j}$ with an arbitrarily large constant. Consequently, we arrive at the necessary conditions $\hat{\rho} \le \sigma_{low}$ and $\hat{\rho} \ge \sigma_{up}/\sqrt{3}$ for comparison to hold. For $\sigma_{low} = 0.1$ and $\sigma_{up} = 0.2$, the numerical test case in Guyon and Henry-Labordère (2011) and Alanko and Avellaneda (2013), these two conditions cannot hold simultaneously, ruling out the possibility of a comparison principle.

Our aim is, thus, to remove the assumption of the comparison principle from Theorem 3.2. The strategy is to consider suitable pairs of supersolutions and subsolutions for which comparison still holds, although the general comparison principle (Comp) may be violated.

To this end, we consider the following 'coupled' pathwise recursion. Given $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G$, $r \in \mathcal{A}_j^F$ and $M \in \mathcal{M}_D$, define the (in general) nonadapted processes $\theta_i^{up} = \theta_i^{up}(\rho^{(1)}, \rho^{(0)}, r, M)$ and $\theta_i^{low} = \theta_i^{low}(\rho^{(1)}, \rho^{(0)}, r, M)$, $i = j, \dots, J$, via the 'pathwise dynamic program'

$$\begin{aligned}
\theta_J^{up} &= \theta_J^{low} = \xi, \\
\theta_i^{up} &= \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \max_{\iota \in \{up, low\}} F_i(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
\theta_i^{low} &= \min_{\iota \in \{up, low\}} G_i\left(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}, \left(r_i^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(r_i^\top \beta_{i+1}\right)_- \theta_{i+1}^{up}\right. \\
&\quad \left. - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right),
\end{aligned} \tag{10}$$

where $\Delta M_i = M_i - M_{i-1}$. We emphasize that these two recursion formulas are coupled, as θ_{i+1}^{low} enters the defining equation for θ_i^{up} and θ_{i+1}^{up} enters the defining equation for θ_i^{low}.

Our main result on the construction of tight upper and lower bounds for the concave-convex dynamic program (2) in absence of the comparison principle now reads as follows:

Theorem 3.5. *Suppose (C) and (R). Then, for every $j = 0, \dots, J$,*

$$\begin{aligned}
Y_j &= \operatorname*{essinf}_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G,\, r \in \mathcal{A}_j^F,\, M \in \mathcal{M}_D} E_j[\theta_j^{up}(\rho^{(1)}, \rho^{(0)}, r, M)] \\
&= \operatorname*{esssup}_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_j^G,\, r \in \mathcal{A}_j^F,\, M \in \mathcal{M}_D} E_j[\theta_j^{low}(\rho^{(1)}, \rho^{(0)}, r, M)], \quad P\text{-a.s.}
\end{aligned}$$

For any $(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_j^G$, $r\in\mathcal{A}_j^F$, *and* $M\in\mathcal{M}_D$, *we have the* P*-almost sure relation*

$$\theta_j^{low}(\rho^{(1)},\rho^{(0)},r,M)\le\theta_j^{up}(\rho^{(1)},\rho^{(0)},r,M)$$

for every $j=0,\dots,J$. *Moreover,*

$$Y_j=\theta_j^{up}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*)=\theta_j^{low}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*)\tag{11}$$

P*-almost surely, for every* $(\rho^{(1,*)},\rho^{(0,*)})\in\mathcal{A}_j^G$ *and* $r^*\in\mathcal{A}_j^F$ *satisfying the duality relations (8)–(9) and with* M^* *being the Doob martingale of* βY.

We finally observe that the processes $E_j[\theta_j^{up}]$ and $E_j[\theta_j^{low}]$ indeed define super- and subsolutions to the dynamic program (2) which are ordered even though the comparison principle may not hold:

Proposition 3.6. *Under the assumptions of Theorem 3.5, the processes* Y^{up} *and* Y^{low} *given by*

$$Y_j^{up}=E_j[\theta_j^{up}(\rho^{(1)},\rho^{(0)},r,M)]\quad\text{and}\quad Y_j^{low}=E_j[\theta_j^{low}(\rho^{(1)},\rho^{(0)},r,M)],\quad j=0,\dots,J$$

are, respectively, super- and subsolutions to (2) for every $(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G$, $r\in A_0^F$, *and* $M\in\mathcal{M}_D$.

4 Relation to information relaxation duals

In this section, we relate a special case of our pathwise dynamic programming approach to the information relaxation dual for a class of two-player dynamic zero-sum games. Throughout the section we assume that, for every $j=0,\dots,J-1$,

$$(v^{(1)}+v^{(0)}u)^\top\beta_{j+1}(\omega)\ge 0\tag{12}$$

for every $\omega\in\Omega$, $(v^{(1)},v^{(0)})\in D_{G_j^\#(\omega,\cdot)}$ and $u\in D_{F_j^\#(\omega,\cdot)}$, where

$$D_{G_j^\#(\omega,\cdot)}=\{(v^{(1)},v^{(0)})\in\mathbb{R}^{D+1},\ G_j^\#(\omega,v^{(1)},v^{(0)})>-\infty\},\ D_{F_j^\#(\omega,\cdot)}=\{u\in\mathbb{R}^D,\ F_j^\#(\omega,u)<+\infty\}$$

denote the domains of $G_j^\#$ and $F_j^\#$. For notational simplicity, we also assume throughout this section that $\mathcal{F}_0$ is a trivial σ-algebra. Without this assumption $E[\cdot]$ must be replaced by $E_0[\cdot]$ below.

We consider, for $j=0,\dots,J$, the multiplicative weights

$$w_j(\omega,v^{(1)},v^{(0)},u)=\prod_{i=0}^{j-1}(v_i^{(1)}+v_i^{(0)}u_i)^\top\beta_{i+1}(\omega)$$

for $\omega\in\Omega$, $v^{(1)}=(v_0^{(1)},...,v_{J-1}^{(1)})\in(\mathbb{R}^D)^J$, $v^{(0)}=(v_0^{(0)},...,v_{J-1}^{(0)})\in\mathbb{R}^J$ and $u=(u_0,...,u_{J-1})\in(\mathbb{R}^D)^J$. Then, Y_0 is the equilibrium value of a stochastic two-player game, namely:

$$Y_0\ =\ \inf_{(\rho^{(1)},\rho^{(0)})\in\mathcal{A}_0^G}\sup_{r\in\mathcal{A}_0^F}E\left[w_J(\rho^{(1)},\rho^{(0)},r)\xi-\sum_{j=0}^{J-1}w_j(\rho^{(1)},\rho^{(0)},r)\left(\rho_j^{(0)}F_j^\#(r_j)+G_j^\#(\rho_j^{(1)},\rho_j^{(0)})\right)\right]$$

$$= \sup_{r \in \mathcal{A}_0^F} \inf_{(\rho^{(1)},\rho^{(0)}) \in \mathcal{A}_0^G} E\left[w_J(\rho^{(1)},\rho^{(0)},r)\xi - \sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},r)\left(\rho_j^{(0)} F_j^{\#}(r_j) + G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\right]$$

(as shown in the appendix). In view of Proposition A.1, the standing assumption (12) of this section guarantees that the comparison principle (Comp) is in force. Hence, we can, by Theorem 3.2, apply the decoupled pathwise programming equations of Example 3.1 in order to construct upper and lower bounds on the equilibrium value Y_0. It turns out (see, again, the appendix for the detailed argument) that, given controls $(\rho^{(1)},\rho^{(0)}) \in \mathcal{A}_0^G$ and a martingale $M \in \mathcal{M}_D$

$$\Theta_0^{up}(\rho^{(1)},\rho^{(0)},M) = \sup_{(u_j)\in\mathbb{R}^D,\, j=0,\ldots,J-1} \Bigg(w_J(\rho^{(1)},\rho^{(0)},u)\xi - \sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},u)\left(\rho_j^{(0)}F_j^{\#}(u_j) + (\rho_j^{(1)} + \rho_j^{(0)}u_j)^\top \Delta M_{j+1} + G_j^{\#}(\rho_j^{(1)},\rho_j^{(0)})\right)\Bigg),$$

which now is a pathwise maximization problem over possibly non-adapted controls u. Notice that the map

$$u \mapsto \sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},u)(\rho_j^{(1)} + \rho_j^{(0)}u_j)^\top \Delta M_{j+1}$$

is a penalty in the sense of Brown et al. (2010), since, for adapted controls $r \in \mathcal{A}_0^F$,

$$\begin{aligned} & E\left[\sum_{j=0}^{J-1} w_j(\rho^{(1)},\rho^{(0)},r)(\rho_j^{(1)} + \rho_j^{(0)}r_j)^\top \Delta M_{j+1}\right] \\ = & \sum_{j=0}^{J-1} E\left[w_j(\rho^{(1)},\rho^{(0)},r)(\rho_j^{(1)} + \rho_j^{(0)}r_j)^\top E_j[\Delta M_{j+1}]\right] = 0 \end{aligned}$$

by the martingale property of M and the tower property of the conditional expectation. Hence in this special case, we can interpret the upper bound $E[\Theta_0^{up}(\rho^{(1)},\rho^{(0)},M)]$ in such a way that, first, one player fixes his strategy $(\rho^{(1)},\rho^{(0)})$ and the penalty by the choice of the martingale M, while, then, the other player is allowed to maximize the penalized problem pathwise. An analogous representation and interpretation holds, of course, for the lower bound $E[\Theta_0^{low}(r,M)]$. Indeed, given a control $r \in \mathcal{A}_0^F$ and a martingale $M \in \mathcal{M}_D$

$$\Theta_0^{low}(r,M) = \inf_{(v_j^{(1)},v_j^{(0)})\in\mathbb{R}^{D+1},\, j=0,\ldots,J-1} \Bigg(w_J(v^{(1)},v^{(0)},r)\xi - \sum_{j=0}^{J-1} w_j(v^{(1)},v^{(0)},r)\left(v_j^{(0)}F_j^{\#}(r_j) + (v_j^{(1)} + v_j^{(0)}r_j)^\top \Delta M_{j+1} + G_j^{\#}(v_j^{(1)},v_j^{(0)})\right)\Bigg),$$

where now the map

$$(v^{(1)},v^{(0)}) \mapsto \sum_{j=0}^{J-1} w_j(v^{(1)},v^{(0)},r)(v_j^{(1)} + v_j^{(0)}r_j)^\top \Delta M_{j+1}$$

is a penalty in the sense of Brown et al. (2010).

In this way, we end up with the information relaxation dual of Brown et al. (2010) for each player given that the other player has fixed his policy and chosen the penalty. This is completely analogous to the recent information relaxation approach to zero-sum games by Haugh and Wang (2015). We emphasize that our choice of a simple subclass of penalties (which contains an optimal penalty by Theorem 3.2) enables us to solve the pathwise optimization problems by the straightforward recursion formulas for Θ^{up} and Θ^{low} given in Example 3.1, while for general choices of the penalties the pathwise maximization problem may often turn out to be a computational bottleneck of the information relaxation approach, compare, e.g., the discussion in Section 4.2 of Brown and Smith (2011) and in Section 2.3 of Haugh and Lim (2012). We also notice that the approach in Haugh and Wang (2015) covers a wider class of zero-sum games than the one described above, but our class of dynamic programming equations extends beyond the setting of stochastic zero-sum games, in particular when the comparison principle is violated (as in Example 3.4). The two approaches are thus complimentary. As an important example, both approaches cover the problem of pricing convertible bonds which (in its simplest form) is given in terms of the dynamic programming equation

$$Y_j = \min\{U_j, \max\{L_j, E_j[Y_{j+1}]\}\}, \quad Y_J = L_J$$

for adapted processes $L_j \leq U_j$. Representations in the sense of pathwise optimal control for this problem were previously studied by Kühn et al. (2007) in continuous time and by Beveridge and Joshi (2011) in discrete time.

5 Applications

In this section, we apply the pathwise dynamic programming methodology to calculate upper and lower bounds for two nonlinear pricing problems, pricing of a long-term interest rate product under bilateral counterparty risk, and option pricing under volatility uncertainty. Traditionally, the valuation of American options was by far the most prominent nonlinear pricing problem both in practice and in academia. In the wake of the financial crisis, various other sources of nonlinearity such as model uncertainty, default risk, liquidity problems or transaction costs have received considerable attention, see the recent monographs Brigo et al. (2013), Crépey (2013), and Guyon and Henry-Labordère (2013).

5.1 Bilateral counterparty risk

Suppose that two counterparties have agreed to exchange a stream of payments C_{t_j} over the sequence of time points $t_0, \ldots, t_J$. For many common interest rate products such as swaps, the sign of C is random – so that the agreed payments may flow in either direction. Therefore, a consistent pricing approach must take into account bilateral default risk, thus introducing nonlinearities into the arising recursive pricing equations which are in general neither convex nor concave. We refer to Crépey et al. (2013) for technical background and an in-depth discussion of the intricacies of pricing under bilateral counterparty risk and funding. We notice that by discretizing their equations (2.14) and (3.8), we arrive at the following nonlinear backward dynamic program for the value of the product Y_j at time t_j (given that there was no default prior to t_j) associated with the payment stream C, namely, $Y_J = C_{t_J}$ and

$$Y_j = (1 - \Delta(r_{t_j} + \gamma_{t_j}(1 - \mathfrak{r})(1 - 2p_{t_j}) + \overline{\lambda}))E_j[Y_{j+1}]$$

$$+\Delta(\gamma_{t_j}(1-\mathfrak{r})(1-3p_{t_j})+\overline{\lambda}-\lambda)E_j[Y_{j+1}]_+ + C_{t_j}.$$

Here E_j denotes the expectation conditional on the market's reference filtration up to time t_j (i.e., the full market information is obtained by enlarging this filtration with the information whether default has yet occurred or not). In focusing on equation (3.8) in Crépey et al. (2013), we consider a pre-default CSA recovery scheme without collateralization, see their paper for background. In the pricing equation, r_t denotes (a proxy to) the risk-less short rate at time t. The rate at which a default of either side occurs at time t is denoted by γ_t. Moreover, p_t is the associated conditional probability that it is the counterparty who defaults, if default occurs at time t. We rule out simultaneous default so that own default happens with conditional probability $1-p_t$, and assume that the three parameters ρ, $\overline{\rho}$ and $\mathfrak{r}$ associated with partial recovery from Crépey et al. (2013) are identical and denoted by $\mathfrak{r}$. Finally, λ and $\overline{\lambda}$ are two constants associated with the costs of external lending and borrowing, and $\Delta = t_{j+1}-t_j$ is the stepsize of the equidistant time partition.

Defining $g_j = 1-\Delta(r_{t_j}+\gamma_{t_j}(1-\mathfrak{r})(1-2p_{t_j})+\overline{\lambda})$ and $h_j = \Delta(\gamma_{t_j}(1-\mathfrak{r})(1-3p_{t_j})+\overline{\lambda}-\lambda)$, we can express the recursion for Y in terms of a concave function G_j and a convex function F_j by setting $F_j(z)=z_+$ and

$$G_j(z,y) = g_j\, z + (h_j)_+\, y - (h_j)_-\, z_+ + C_{t_j},$$

so that $Y_j = G_j(E_j[Y_{j+1}], F_j(E_j[Y_{j+1}]))$. Denote by $(\tilde{Y}_j)$ a numerical approximation of the process (Y_j), by $(\tilde{Q}_j)$ a numerical approximation of $(E_j[Y_{j+1}])$ and by $(\tilde{M}_j)$ a martingale which we think of as an approximation to the Doob martingale of Y. In terms of these inputs, the pathwise recursions (10) for upper and lower bound are given by

$$\theta_j^{up} \;=\; \left(\tilde{\rho}_j^{(1)}\right)_+\left(\theta_{j+1}^{up}-\Delta\tilde{M}_{j+1}\right)-\left(\tilde{\rho}_j^{(1)}\right)_-\left(\theta_{j+1}^{low}-\Delta\tilde{M}_{j+1}\right)+\tilde{\rho}_j^{(0)}\left(\theta_{j+1}^{up}-\Delta\tilde{M}_{j+1}\right)_+ + C_{t_j}$$

and

$$\theta_j^{low} = \min_{\iota\in\{up,low\}} g_j\left(\theta_{j+1}^{\iota}-\Delta\tilde{M}_{j+1}\right)-(h_j)_-\left(\theta_{j+1}^{\iota}-\Delta\tilde{M}_{j+1}\right)_+ + (h_j)_+\,\tilde{s}_j\left(\theta_{j+1}^{low}-\Delta\tilde{M}_{j+1}\right)+C_{t_j}$$

where

$$(\tilde{\rho}_j^{(1)},\tilde{\rho}_j^{(0)},\tilde{s}_j) = \begin{cases}(g_j,(h_j)_+,0)\,, & \tilde{Q}_j<0,\\ (g_j-(h_j)_-,(h_j)_+,1)\,, & \tilde{Q}_j\geq 0.\end{cases}$$

For the payment stream C_{t_j}, we consider a swap with notional N, fixed rate R and an equidistant sequence of tenor dates $\mathcal{T}=\{T_0,\ldots,T_K\}\subseteq\{t_0,\ldots t_J\}$. Denote by δ the length of the time interval between T_i and T_{i+1} and by $P(T_{i-1},T_i)$ the T_{i-1}-price of a zero-bond with maturity T_i. Then, the payment process C_{t_j} is given by

$$C_{T_i} = N\cdot\left(\frac{1}{P(T_{i-1},T_i)}-(1+R\delta)\right)$$

for $T_i\in\mathcal{T}\setminus\{T_0\}$ and $C_{t_j}=0$ otherwise, see Brigo and Mercurio (2006), Chapter 1.

For r and γ, we implement the model of Brigo and Pallavicini (2007), assuming that the risk-neutral dynamics of r is given by a two-factor Gaussian short rate model, a reparametrization of the two-factor Hull-White model, while γ is a CIR process. For the conditional default probabilities p_t we assume $p_t = 0 \wedge \tilde{p}_t \vee 1$ where $\tilde{p}$ is an Ornstein-Uhlenbeck process. In continuous time, this corresponds to the system of stochastic differential equations

$$dx_t = -\kappa_x x_t dt + \sigma_x dW_t^x, \;\; dy_t = -\kappa_y y_t dt + \sigma_y dW_t^y$$
$$d\gamma_t = \kappa_\gamma(\mu_\gamma - \gamma_t)dt + \sigma_\gamma \sqrt{\gamma_t} dW_t^\gamma, \;\; d\tilde{p}_t = \kappa_p(\mu_p - \tilde{p}_t)dt + \sigma_p dW_t^p$$

with $r_t = r_0 + x_t + y_t$, $x_0 = y_0 = 0$. Here, W^x, W^y and W^γ are Brownian motions with instantaneous correlations ρ_{xy}, $\rho_{x\gamma}$ and $\rho_{y\gamma}$. In addition, we assume that $W_t^p = \rho_{\gamma p} W_t^\gamma + \sqrt{1-\rho_{\gamma p}^2} W_t$ where the Brownian motion W is independent of (W^x, W^y, W^γ). We choose the filtration generated by the four Brownian motions as the reference filtration.

For the dynamics of x, y and $\tilde{p}$, exact time discretizations are available in closed form. We discretize γ using the fully truncated scheme of Lord et al. (2010). The bond prices $P(t,s)$ are given as an explicit function of x_t and y_t in this model. This implies that the swap's "clean price", i.e., the price in the absence of counterparty risk is given in closed form as well, see Sections 1.5 and 4.2 of Brigo and Mercurio (2006).

We consider 60 half-yearly payments over a horizon of $T = 30$ years, i.e., $\delta = 0.5$. J is always chosen as an integer multiple of 60 so that δ is an integer multiple of $\Delta = T/J$. For the model parameters, we choose

$$(r_0, \kappa_x, \sigma_x, \kappa_y, \sigma_y) = (0.03, 0.0558, 0.0093, 0.5493, 0.0138),$$
$$(\gamma_0, \mu_\gamma, \kappa_\gamma, \sigma_\gamma, p_0, \mu_p, \kappa_p, \sigma_p) = (0.0165,\ 0.026,\ 0.4, 0.14,\ 0.5,\ 0.5,\ 0.8,\ 0.2),$$
$$(\rho_{xy}, \rho_{x\gamma}, \rho_{y\gamma}, \mathfrak{r}, \lambda, \overline{\lambda}, N) = (-0.7,\ 0.05,\ -0.7,\ 0.4,\ 0.015,\ 0.045,\ 1).$$

We thus largely follow Brigo and Pallavicini (2007) for the parametrization of r and γ but leave out their calibration to initial market data and choose slightly different correlations to avoid the extreme cases of a perfect correlation or independence of r and γ. The remaining parameters J, R and $\rho_{\gamma p}$ are varied in the numerical experiments below.

Except for the different recursions for θ^{up} and θ^{low}, the overall numerical procedure is similar to the one in Bender et al. (2015), so we refer to that paper for a more detailed exposition. We calculate the initial approximation $(\tilde{Q}_j, \tilde{Y}_j)$ by a least-squares Monte Carlo approach. We first simulate $N_r = 10^5$ trajectories of $(x_{t_j}, y_{t_j}, \gamma_{t_j}, p_{t_j}, C_{t_j})$ forward in time, the so-called regression paths. Then, for $j = J-1, \ldots 0$, we compute $\tilde{Q}_j$ by least-squares regression of $\tilde{Y}_{j+1}$ onto a set of basis functions. As the four basis functions, we use 1, γ_{t_j}, $\gamma_{t_j} \cdot p_{t_j}$ and the closed-form expression for the clean price at time t_j of the swap's remaining payment stream. $\tilde{Y}_j$ is computed by application of F_j and G_j to $\tilde{Q}_j$. Storing the coefficients from the regression, we next simulate $N_o = 10^6$ independent trajectories of $(x_{t_j}, y_{t_j}, \gamma_{t_j}, p_{t_j}, C_{t_j}, \tilde{Q}_j, \tilde{Y}_j)$ forward in time, the so-called outer paths. In order to simulate an approximate Doob martingale $\tilde{M}$ of $\tilde{Y}_j$, we rely on a subsampling approach as proposed in Andersen and Broadie (2004). Along each outer path and for all j, we simulate $N_i = 100$ copies of $\tilde{Y}_{j+1}$ conditional on the observed value of $(x_{t_j}, y_{t_j}, \gamma_{t_j}, \tilde{p}_{t_j})$ and take the average of these copies to obtain an unbiased estimator of $E_j[\tilde{Y}_{j+1}]$.

After these preparations, we can go through the recursion for θ^{up} and θ^{low} along each outer path. Denote by $\hat{Y}_0^{up}$ and $\hat{Y}_0^{low}$ the resulting Monte Carlo estimators of $E[\theta_0^{up}]$ and $E[\theta_0^{low}]$ and their associated empirical standard deviations by $\hat{\sigma}^{up}$ and $\hat{\sigma}^{low}$. Then, an asymptotic 95%-confidence interval for Y_0 is given by

$$[\hat{Y}_0^{low} - 1.96\hat{\sigma}^{low},\ \hat{Y}_0^{up} + 1.96\hat{\sigma}^{up}]$$

J	Clean Price	$\rho_{\gamma p} = 0.8$		$\rho_{\gamma p} = 0$		$\rho_{\gamma p} = -0.8$	
120	0	21.53 (0.02)	21.70 (0.02)	25.13 (0.02)	25.31 (0.02)	28.42 (0.02)	28.64 (0.02)
360	0	21.34 (0.02)	21.52 (0.02)	24.97 (0.02)	25.16 (0.02)	28.25 (0.02)	28.50 (0.02)
720	0	21.28 (0.02)	21.46 (0.02)	24.88 (0.02)	25.07 (0.02)	28.16 (0.02)	28.41 (0.02)
1440	0	21.28 (0.02)	21.46 (0.02)	24.88 (0.02)	25.08 (0.02)	28.15 (0.02)	28.41 (0.02)

Table 1: Lower and upper bound estimators for varying values of $\rho_{\gamma p}$ and J with $R = 275.12$ basis points (b.p.), $N_o = 10^6$ and $N_i = 100$. Prices and standard deviations (in brackets) are given in b.p.

N_i	Clean Price	$\rho_{\gamma p} = 0.8$		$\rho_{\gamma p} = 0$		$\rho_{\gamma p} = -0.8$	
10	0	21.14 (0.17)	22.15 (0.17)	24.79 (0.17)	25.81 (0.17)	28.09 (0.17)	29.23 (0.17)
20	0	21.38 (0.12)	21.94 (0.12)	25.01 (0.12)	25.58 (0.12)	28.30 (0.12)	28.96 (0.12)
50	0	21.28 (0.09)	21.57 (0.09)	24.90 (0.09)	25.19 (0.09)	28.19 (0.09)	28.54 (0.09)
100	0	21.22 (0.07)	21.41 (0.07)	24.83 (0.07)	25.02 (0.07)	28.10 (0.07)	28.35 (0.07)
200	0	21.42 (0.06)	21.55 (0.06)	25.00 (0.06)	25.14 (0.06)	28.29 (0.06)	28.48 (0.06)

Table 2: Lower and upper bound estimators for varying values of $\rho_{\gamma p}$ and N_i with $N_o = 10^5$, $R = 275.12$ b.p. and $J = 720$. Prices and standard deviations (in brackets) are given in b.p.

Table 1 displays upper and lower bound estimators with their standard deviations for different values of the time discretization and the correlation between γ and p. Here, R is chosen as the fair swap rate in the absence of default risk, i.e., it is chosen such that the swap's clean price at $j = 0$ is zero. The four choices of J correspond to a quarterly, monthly, bi-weekly, and weekly time discretization, respectively. In all cases, the width of the resulting confidence interval is about 1.2% of the value. While variations between the two finer time discretizations can be explained by standard deviations, the two rougher discretizations lead to values which are slightly higher. We thus work with the bi-weekly discretization $J = 720$ in the following. The effect of varying the correlation parameter of γ and p also has the expected direction. Roughly, if $\rho_{\gamma p}$ is positive then larger values of the overall default rate go together with larger conditional default risk of the counterparty and smaller conditional default risk of the party, making the product less valuable to the party. While this effect is not as pronounced as the overall deviation from the clean price, the bounds are easily tight enough to differentiate between the three cases.

Table 2 shows the effect of varying the number of inner paths in the same setting. Since a larger number of inner paths improves the approximation quality of the martingale, and since we have pathwise optimality for the limiting Doob martingale, we observe two effects, a reduction in standard deviations and a reduction of the distance between lower and upper bound estimators. To display these effects more clearly, we have decreased the number of outer paths to $N_o = 10^5$ in this table, leading to the expected increase in the overall level of standard deviations.

Finally, Table 3 displays the adjusted fair swap rates accounting for counterparty risk and

$\rho_{\gamma p}$	Adjusted Fair Swap Rate	Clean Price	Bounds	
0.8	290.91	-31.70	-0.07 (0.07)	0.12 (0.07)
0	293.75	-37.41	-0.09 (0.07)	0.12 (0.07)
-0.8	296.41	-42.75	-0.12 (0.07)	0.14 (0.07)

Table 3: Adjusted fair swap rates and lower and upper bound estimators for varying values of $\rho_{\gamma p}$ with $N_o = 10^5$, $N_i = 100$ and $J = 720$. Rates, prices and standard deviations (in brackets) are given in b.p.

funding for the three values of $\rho_{\gamma p}$, i.e., the values of R which set the adjusted price to zero in the three scenarios. To identify these rates, we fix a set of outer, inner and regression paths and define $\mu(R)$ as the midpoint of the confidence interval we obtain when running the algorithm with these paths and rate R for the fixed leg of the swap. We apply a standard bisection method to find the zero of $\mu(R)$. The confidence intervals for the prices in Table 3 are then obtained by validating these swap rates with a new set of outer and inner paths. We observe that switching from a clean valuation to the adjusted valuation with $\rho = 0.8$ increases the fair swap rate by 16 basis points (from 275 to 291). Changing ρ from 0.8 to -0.8 leads to a further increase by 5 basis points.

5.2 Uncertain Volatility Model

In this section, we study in more detail the uncertain volatility model (UVM) from Example 3.4, which goes back to Avellaneda et al. (1995) and Lyons (1995). After introducing the setting and motivating the time discretization, we discuss the implementation of our algorithm and present our numerical results.

There is a risky asset X whose dynamics (in discounted units) under a risk-neutral measure is given by the stochastic differential equation

$$dX_t = \sigma_t X_t dW_t, \quad X_0 = x_0 \in \mathbb{R}_+,$$

where W is a Brownian motion and the (uncertain) progressively measurable (with respect to the information $\mathcal{F}_t$, $0 \le t \le T$, generated by the Brownian motion up to time t) process σ takes values in the closed interval $[\sigma_{low}, \sigma_{up}]$ with $0 < \sigma_{low} < \sigma_{up} < \infty$. We wish to compute upper and lower bounds on the (worst case) price of a European option with maturity $T \in \mathbb{R}_+$ and payoff $g(X_T)$ under uncertain volatility, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with linear growth. This price is given by

$$Y_0 = \sup_\sigma E[g(X_T^\sigma)], \tag{13}$$

where the supremum runs over the set of progressively measurable processes σ taking values in $[\sigma_{low}, \sigma_{up}]$ and X_T^σ is the time-T-value of the process X under a fixed volatility process σ. As this is a stochastic control problem in a Markovian setting, it is well known that the value of the option can be represented by the solution of the Hamilton-Jacobi-Bellman equation. Following Avellaneda et al. (1995), the Hamilton-Jacobi-Bellman equation for this problem is

$$-u_t(t,x) - \frac{1}{2}x^2(\sigma_{up}^2(u_{xx}(t,x))_+ - \sigma_{low}^2(u_{xx}(t,x))_-) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}_+ \tag{14}$$

$$u(T,x) = g(x), \quad x \in \mathbb{R}_+.$$

This fully nonlinear second-order parabolic partial differential equation is known as the Black-Scholes-Barenblatt equation.

The discretization scheme in Example 3.4 was suggested by Guyon and Henry-Labordère (2011). In order to recognize that it is a special case of the time discretization scheme for a larger class of fully nonlinear second-order parabolic partial differential equations as introduced and analyzed by Fahim et al. (2011), we make the following transformation in the space-variable: Fix $\hat{\rho} \in \mathbb{R}_+$ and define

$$v(t,x) = u(t, x_0 e^{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 t}).$$

Equation (14) then becomes

$$-v_t(t,x) - \frac{1}{2}v_{xx}(t,x) - \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)(v_{xx}(t,x) - \hat{\rho} v_x(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \tag{15}$$

$$v(T,x) = g(x_0 e^{\hat{\rho}x - \frac{1}{2}\hat{\rho}^2 T}), \quad x \in \mathbb{R}.$$

Under our assumptions, there exists a unique smooth solution to (15) (satisfying appropriate growth conditions), see Pham (2009). Let $0 = t_0 < t_1 < \ldots < t_J = T$ be an equidistant partition of the interval $[0,T]$, where $J \in \mathbb{N}$, and set $\Delta = t_j - t_{j-1}$ and $\Delta W_j = W_{t_j} - W_{t_{j-1}}$. Now, following the arguments in Fahim et al. (2011), we apply Itô's formula and take conditional expectation with respect to $\mathcal{F}_{t_j}$ (denoted by E_j) in order to obtain, by (15),

$$\begin{aligned}
v(t_j, W_{t_j}) &= E_j\left[v(t_{j+1}, W_{t_{j+1}}) - \int_{t_j}^{t_{j+1}} \left(v_t(s,W_s) + \frac{1}{2}v_{xx}(s,W_s)\right) ds\right] \\
&= E_j\left[v(t_{j+1}, W_{t_{j+1}}) + \int_{t_j}^{t_{j+1}} \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)(v_{xx}(s,W_s) - \hat{\rho} v_x(s,W_s))\, ds\right] \\
&\approx E_j\left[v(t_{j+1}, W_{t_{j+1}})\right] + \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)\left(v_{xx}(t_j, W_{t_j}) - \hat{\rho} v_x(t_j, W_{t_j})\right)\Delta
\end{aligned}$$

Applying Malliavin Monte Carlo weights for the first and second derivative (see, e.g., Fahim et al., 2011, Lemma 2.1), we arrive at the following discrete time dynamic programming equation:

$$\begin{aligned}
Y_J &= g(X_T^{\hat{\rho}}), \\
Y_j &= E_j[Y_{j+1}] + \max_{\sigma \in \{\sigma_{low}, \sigma_{up}\}} \frac{1}{2}\left(\frac{\sigma^2}{\hat{\rho}^2} - 1\right)\Gamma_j \Delta, \\
\Gamma_j &= E_j\left[B_{j+1}^{\hat{\rho}} Y_{j+1}\right], \quad \text{where} \quad B_{j+1}^{\hat{\rho}} = \frac{\Delta W_{j+1}^2}{\Delta^2} - \hat{\rho}\frac{\Delta W_{j+1}}{\Delta} - \frac{1}{\Delta}
\end{aligned} \tag{16}$$

and $X_T^{\hat{\rho}}$ is the value of X_T under the constant reference volatility $\hat{\rho}$.

Notice that the reference volatility $\hat{\rho}$ is a choice parameter in the discretization. The basic idea is to view the uncertain volatility model as a suitable correction of a Black-Scholes model with volatility $\hat{\rho}$. Let $G_j(z,y) = y$, $s_\iota = \frac{1}{2}(\frac{\sigma_\iota^2}{\hat{\rho}^2} - 1)$ for $\iota \in \{up, low\}$, and

$$F_j(z) = z^{(1)} + \max_{s \in \{s_{low}, s_{up}\}} s z^{(2)} \Delta,$$

where $z^{(i)}$ denotes the i-th component of the two-dimensional vector z, and define the $\mathbb{R}^2$-valued process β by

$$\beta_j = \begin{pmatrix} 1 \\ B_j^{\hat{\rho}} \end{pmatrix}, \quad j = 1, \ldots, J.$$

With these choices, the dynamic programming equation for Y in (16) is of the form (2).

As in Section 5.1, we denote in the following by $(\tilde{Y}_j)$ a numerical approximation of the process (Y_j), by $(\tilde{Q}_j)$ a numerical approximation of $(E_j[Y_{j+1}])$ and by $(\tilde{\Gamma}_j)$ a numerical approximation of (Γ_j). Furthermore, $(\tilde{M}_j) = (\tilde{M}_j^{(1)}, \tilde{M}_j^{(2)})^\top$ denotes an approximation of the Doob martingale M of $\beta_j Y_j$, which is given by

$$M_{j+1} - M_j = \begin{pmatrix} Y_{j+1} - E_j[Y_{j+1}] \\ B_{j+1}^{\hat{\rho}} Y_{j+1} - E_j\left[B_{j+1}^{\hat{\rho}} Y_{j+1}\right] \end{pmatrix}.$$

The recursion (10) for θ_j^{up} and θ_j^{low} which is based on these input approximations can be written as

$$\theta_j^{up} = \max_{\iota \in \{up, low\}} \max_{s \in \{s_{low}, s_{up}\}} \left\{ \theta_{j+1}^{\iota} - \Delta \tilde{M}_{j+1}^{(1)} + s B_{j+1}^{\hat{\rho}} \theta_{j+1}^{\iota} \Delta - s \Delta \tilde{M}_{j+1}^{(2)} \Delta \right\}$$

and

$$\theta_j^{low} = \left(\tilde{r}_j^{(1)} + \tilde{r}_j^{(2)} B_{j+1}^{\hat{\rho}} \Delta\right)_+ \theta_{j+1}^{low} - \left(\tilde{r}_j^{(1)} + \tilde{r}_j^{(2)} B_{j+1}^{\hat{\rho}} \Delta\right)_- \theta_{j+1}^{up} - \tilde{r}_j^{(1)} \Delta \tilde{M}_{j+1}^{(1)} - \tilde{r}_j^{(2)} \Delta \tilde{M}_{j+1}^{(2)} \Delta,$$

where $\tilde{r}_j = (\tilde{r}_j^{(1)}, \tilde{r}_j^{(2)})$ is given by

$$\tilde{r}_j = \begin{cases} (1, s_{low}), & \tilde{\Gamma}_j < 0 \\ (1, s_{up}), & \tilde{\Gamma}_j \geq 0. \end{cases}$$

For the payoff, we consider a European call-spread option with strikes K_1 and K_2, i.e.,

$$g(x) = (x - K_1)_+ - (x - K_2)_+,$$

which is also studied in Guyon and Henry-Labordère (2011) and Alanko and Avellaneda (2013). Following them, we choose the maturity $T = 1$ as well as $K_1 = 90$ and $K_2 = 110$ and $x_0 = 100$. The parameters $\hat{\rho}$, σ_{low} and σ_{up} are varied in the numerical experiments below.

In order to calculate the initial approximations $(\tilde{Y}_j)$ and $(\tilde{\Gamma}_j)$, we apply the martingale basis variant of least-squares Monte Carlo which was proposed in Bender and Steiner (2012) in the context of backward stochastic differential equations and dates back to Glasserman and Yu (2004) for the optimal stopping problem. The basic idea is to use basis functions whose conditional expectations (under the reference Black-Scholes model) can be calculated in closed form. When $\tilde{Y}_{j+1}$ is given as a linear combination of such basis functions, $E_j[\tilde{Y}_{j+1}]$ can be calculated without regression by simply applying the same coefficients to the conditional expectations of the basis functions. Similarly the Gamma of $\tilde{Y}$ can be represented in terms of the second derivatives of the basis functions. To be more precise, we fix K basis functions $\eta_{J,k}(x)$, $k = 1, \ldots, K$, at terminal time and assume that

$$\eta_{j,k}(x) = E[\eta_{J,k}(X_J^{\hat{\rho}}) | X_j^{\hat{\rho}} = x]$$

can be computed in closed form and is twice differentiable in x. Then, it is straightforward to check that

$$\begin{aligned} E[\eta_{j+1,k}(X_{j+1}^{\hat{\rho}})|X_j^{\hat{\rho}} = x] &= \eta_{j,k}(x) \\ E\left[B_{j+1}^{\hat{\rho}} \eta_{j+1,k}(X_{j+1}^{\hat{\rho}}) \middle| X_j^{\hat{\rho}} = x \right] &= \hat{\rho}^2 x^2 \frac{d^2}{dx^2} \eta_{j,k}(x). \end{aligned}$$

For a given approximation of $\tilde{Y}_{j+1}$, $j = 0, \ldots, J-1$, in terms of such martingale basis functions, a first approximation of $\tilde{Y}_j$ can be calculated without regression by computing the right hand side of (16) explicitly. This approximation is regressed onto the basis functions to ensure that $\tilde{Y}_j$ is again a linear combination of basis functions. This overall approach has two key advantages. First, since $\tilde{Q}_j = E_j[\tilde{Y}_{j+1}]$ and $\tilde{\Gamma}_j$ are exact conditional expectations of $\tilde{Y}_{j+1}$ and $B_{j+1}^{\hat{\rho}} \tilde{Y}_{j+1}$, the martingale

$$\tilde{M}_{j+1} - \tilde{M}_j = \begin{pmatrix} \tilde{Y}_{j+1} - \tilde{Q}_j \\ B_{j+1}^{\hat{\rho}} \tilde{Y}_{j+1} - \tilde{\Gamma}_j \end{pmatrix}$$

can be computed without subsimulations, and thus without the associated sampling error and simulation costs. Second, since we calculate $\tilde{\Gamma}_j$ by twice differentiating the Y-approximation, we avoid to use the simulated Γ-weights $B_{j+1}^{\hat{\rho}}$ in the regression. Due to the factor $\Delta W_{j+1}^2 \cdot \Delta^{-2}$, these weights are otherwise a source of large standard errors. See Alanko and Avellaneda (2013) for a discussion of this problem and some control variates (which are not needed in our approach).

We first simulate $N_r = 10^5$ regression paths of the process $(X_j^{\hat{\rho}})$ under the constant volatility $\hat{\rho}$. For the regression, we do not start all paths at x_0 but rather start $N_r/200$ trajectories at each of the points $31, \ldots, 230$. Since X is a geometric Brownian motion under $\hat{\rho}$, it can be simulated exactly. Starting the regression paths at multiple points allows to reduce the instability of regression coefficients arising at early time points. See Rasmussen (2005) for a discussion of this stability problem and of the method of multiple starting points. For the computation of $\tilde{Y}_j$ and $\tilde{Q}_j$ we choose 162 basis functions. The first three are 1, x and $E[g(X_J^{\hat{\rho}})|X_j^{\hat{\rho}} = x]$. Note that the third one is simply the Black-Scholes price (under $\hat{\rho}$) of the spread option g. For the remaining 159 basis functions, we also choose Black-Scholes prices of spread options with respective strikes $K^{(k)}$ and $K^{(k+1)}$ for $k = 1, \ldots, 159$, where the numbers $K^{(1)}, \ldots, K^{(160)}$ increase from 20.5 to 230.5. The second derivatives of these basis functions are just (differences of) Black-Scholes Gammas.

Like in Section 5.1, we then simulate $(X_j^{\hat{\rho}}, \tilde{Y}_j, \tilde{Q}_j, \tilde{\Gamma}_j)$ along $N_o = 10^5$ outer paths started in x_0. As pointed out above, the Doob martingale of $\beta_j \tilde{Y}_j$ is available in closed form in this case. Finally, the recursions for θ^{up} and θ^{low} are calculated backwards in time on each outer path.

As a first example we consider $\sigma_{low} = 0.1$ and $\sigma_{up} = 0.2$ as in Guyon and Henry-Labordère (2011) and Alanko and Avellaneda (2013). Following Vanden (2006), the continuous-time limiting price of a European call-spread option in this setting is given by 11.2046. Table 4 shows the approximated prices $\tilde{Y}_0$ as well as upper and lower bounds for $\hat{\rho} = \frac{0.2}{\sqrt{3}} \approx 0.115$ depending on the time discretization. This is the smallest choice of $\hat{\rho}$, for which the monotonicity condition in Theorem 3.3 can only be violated when the absolute values of the Brownian increments are large, cp. Example 3.4. As before, $\hat{Y}_0^{up}$ and $\hat{Y}_0^{low}$ denote Monte Carlo estimates of $E[\theta_0^{up}]$ respectively $E[\theta_0^{low}]$. The numerical results suggest convergence from below towards the continuous-time limit for finer time discretizations. This is intuitive in this example, since finer time discretizations allow for richer choices of the process (σ_t) in the maximization problem (13). We notice that the bounds are fairly tight (with, e.g., a relative width of 1.3% for the 95% confidence interval with $J = 21$ time discretization points), although the upper bound begins to deteriorate as $\tilde{Y}_0$

approaches its limiting value. The impact of increasing $\hat{\rho}$ to 0.15 (as proposed in Guyon and Henry-Labordère, 2011; Alanko and Avellaneda, 2013) is shown in Table 5. The relative width of the 95%-confidence interval is now about 0.6 % for up to $J = 35$ time steps, but also the convergence to the continuous time limit appears to be slower with this choice of $\hat{\rho}$.

J	3	6	9	12	15	18	21	24
$\tilde{Y}_0$	10.8553	11.0500	11.1054	11.1340	11.1484	11.1585	11.1666	11.1710
$\hat{Y}_0^{low}$	10.8550 (0.0001)	11.0502 (0.0002)	11.1060 (0.0005)	11.1344 (0.0002)	11.1486 (0.0002)	11.1585 (0.0006)	11.1666 (0.0003)	11.1683 (0.0032)
$\hat{Y}_0^{up}$	10.8591 (0.0001)	11.0536 (0.0002)	11.1116 (0.0005)	11.1438 (0.0007)	11.1728 (0.0056)	11.2097 (0.0088)	11.2764 (0.0173)	11.5593 (0.0984)

Table 4: Approximated price as well as lower and upper bounds for $\hat{\rho} = \frac{0.2}{\sqrt{3}}$ for different time discretizations. Standard deviations are given in brackets

J	5	10	15	20	25	30	35	40
$\tilde{Y}_0$	10.8153	10.9982	11.0684	11.1027	11.1241	11.1386	11.1479	11.1554
$\hat{Y}_0^{low}$	10.8153 (0.0001)	10.9983 (0.0001)	11.0683 (0.0001)	11.1023 (0.0001)	11.1237 (0.0001)	11.1376 (0.0002)	11.1462 (0.0002)	11.0101 (0.1391)
$\hat{Y}_0^{up}$	10.8167 (0.0001)	11.0028 (0.0001)	11.0728 (0.0001)	11.1092 (0.0002)	11.1379 (0.0008)	11.1687 (0.0030)	11.2058 (0.0047)	12.0483 (0.6464)

Table 5: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.15$ for different time discretizations. Standard deviations are given in brackets

Comparing Table 5 with the results in Alanko and Avellaneda (2013), we observe that their point estimates for Y_0 at time discretization levels $J = 10$ and $J = 20$ do not lie in our confidence intervals which are given by $[10.9981, 11.0030]$ and $[11.1021, 11.1096]$, indicating that their least-squares Monte Carlo estimator may still suffer from large variances (although they apply control variates). The dependence of the time discretization error on the choice of the reference volatility $\hat{\rho}$ is further illustrated in Table 6, which displays the mean and the standard deviation of 30 runs of the martingale basis algorithm for different choices of $\hat{\rho}$ and up to 640 time steps. By and large, convergence is faster for smaller choices of $\hat{\rho}$, but the algorithm becomes unstable when the reference volatility is too small.

J	10	20	40	80	160	320	640
$\hat{\rho} = 0.06$	79.7561 (5.1739)	$1.6421 \cdot 10^5$ ($8.4594 \cdot 10^5$)	$1.7010 \cdot 10^{16}$ ($6.3043 \cdot 10^{16}$)	$3.2151 \cdot 10^{24}$ ($1.7603 \cdot 10^{25}$)	$1.8613 \cdot 10^{24}$ ($7.0234 \cdot 10^{24}$)	$4.5672 \cdot 10^{39}$ ($2.5016 \cdot 10^{40}$)	$7.0277 \cdot 10^{39}$ ($3.7590 \cdot 10^{40}$)
$\hat{\rho} = 0.08$	11.6463 (0.2634)	12.3183 (1.5447)	130.2723 (372.2625)	11.8494 (1.1846)	11.6951 (1.5772)	$5.4389 \cdot 10^3$ ($1.8766 \cdot 10^4$)	$1.0153 \cdot 10^8$ ($3.5571 \cdot 10^8$)
$\hat{\rho} = 0.1$	11.1552 (0.0031)	11.1823 (0.0040)	11.1942 (0.0005)	11.1999 (0.0002)	11.2026 (0.0001)	11.2047 (0.0001)	11.2057 (0.0001)
$\hat{\rho} = 0.15$	10.9985 (0.0006)	11.1027 (0.0006)	11.1556 (0.0004)	11.1821 (0.0003)	11.1952 (0.0003)	11.2017 (0.0003)	11.2047 (0.0002)
$\hat{\rho} = 0.2$	10.7999 (0.0007)	10.9746 (0.0005)	11.0819 (0.0005)	11.1455 (0.0006)	11.1802 (0.0005)	11.1980 (0.0006)	11.2073 (0.0006)
$\hat{\rho} = 0.5$	9.7088 (0.0004)	9.9652 (0.0005)	10.2306 (0.0009)	10.4945 (0.0015)	10.7453 (0.0047)	10.9635 (0.0076)	11.1248 (0.0103)

Table 6: Mean of $L = 30$ simulations of $\tilde{Y}_0$ for different $\hat{\rho}$ and discretizations. Standard deviations are given in brackets.

In order to gain a better understanding of how the performance of the method depends on the input parameters, we also consider the case $\sigma_{low} = 0.3$ and $\sigma_{up} = 0.4$. Following Vanden (2006), the price of the European call-spread option in the continuous-time limit is 9.7906 in this case. We get qualitatively the same results as for the previous example, in the sense that convergence is faster for the smaller reference volatility and that the upper bound estimators begin to deteriorate as the time partition becomes too fine. However, quantitatively, the numerical results are better than in the previous example (confidence intervals remain tight for finer time partitions, for which the discretized price has a relative error of about 0.002 compared to the continuous time limit). This is quite likely to be connected to the fact that the ratio between σ_{up} and σ_{low} is smaller in this second example.

J	3	6	9	12	15	18	21	24	27	30
$\tilde{Y}_0$	9.6206	9.7147	9.7435	9.7574	9.7643	9.7683	9.7717	9.7750	9.7767	9.7772
$\hat{Y}_0^{low}$	9.6139 (0.0001)	9.7147 (0.0002)	9.7433 (0.0002)	9.7573 (0.0003)	9.7642 (0.0002)	9.7681 (0.0005)	9.7728 (0.0006)	9.7740 (0.0012)	9.7768 (0.0002)	9.7769 (0.0002)
$\hat{Y}_0^{up}$	9.6211 (0.0002)	9.7167 (0.0002)	9.7475 (0.0006)	9.7629 (0.0005)	9.7725 (0.0016)	9.7848 (0.0028)	9.7933 (0.0046)	9.8010 (0.0052)	9.8248 (0.0115)	9.9354 (0.0592)

Table 7: Approximated price as well as lower and upper bounds for $\hat{\rho} = \frac{\sigma_{up}}{\sqrt{3}}$ for different time discretizations. Standard deviations are given in brackets

J	10	20	30	40	50	60	70	80
$\tilde{Y}_0$	9.6061	9.6924	9.7235	9.7409	9.7509	9.7574	9.7626	9.7653
$\hat{Y}_0^{low}$	9.6060 (0.0001)	9.6922 (<0.0001)	9.7231 (<0.0001)	9.7406 (<0.0001)	9.7506 (<0.0001)	9.7571 (0.0001)	9.7622 (0.0001)	9.7644 (0.0002)
$\hat{Y}_0^{up}$	9.6063 (0.0001)	9.6930 (<0.0001)	9.7248 (0.0001)	9.7435 (0.0001)	9.7567 (0.0002)	9.7707 (0.0006)	9.7946 (0.0018)	9.8651 (0.0088)

Table 8: Approximated price as well as lower and upper bounds for $\hat{\rho} = 0.35$ for different time discretizations. Standard deviations are given in brackets

6 Conclusion

This paper proposes a new method for constructing high-biased and low-biased Monte Carlo estimators for solutions of stochastic dynamic programming equations. When applied to the optimal stopping problem, the method simplifies to the classical primal-dual approach of Rogers (2002) and Haugh and Kogan (2004) (except that the martingale penalty appears as a control variate in the lower bound). Our approach is complementary to earlier generalizations of this methodology by Rogers (2007) and Brown et al. (2010) whose approaches start out from a primal optimization problem rather than from a dynamic programming equation. The resulting representation of high-biased and low-biased estimators in terms of a pathwise recursion makes the method very tractable from a computational point of view. Suitably coupling upper and lower bounds in the recursion enables us to handle situations which are outside the scope of classical primal-dual approaches, because the dynamic programming equation fails to satisfy a comparison principle.

A Proofs

A.1 Proof of Lemma 2.2

Let $\overline{Z} \in L_j^{\infty-}(\mathbb{R}^N)$. Notice first that, since f_j is convex and closed, we have $f_j^{\#\#} = f_j$ by Theorem 12.2 in Rockafellar (1970) and thus

$$f_j(\overline{Z}) \geq \rho^\top \overline{Z} - f_j^{\#}(\rho) \tag{17}$$

holds ω-wise for any random variable ρ. We next show that there exists an $\mathcal{F}_j$-measurable random variable $\overline{\rho}_j$ for which (17) holds with P-almost sure equality. To this end, we apply Theorem 7.4 in Cheridito et al. (2015) which yields the existence of an $\mathcal{F}_j$-measurable subgradient to f_j, i.e., existence of an $\mathcal{F}_j$-measurable random variable $\overline{\rho}_j$ such that for all $\mathcal{F}_j$-measurable $\mathbb{R}^N$-valued random variables Z

$$f_j\left(\overline{Z} + Z\right) - f_j\left(\overline{Z}\right) \geq \overline{\rho}_j^\top Z, \quad P\text{-a.s.} \tag{18}$$

From (18) (with the choice $Z = z - \bar{Z}$ for $z \in \mathbb{Q}^N$), we conclude that

$$\overline{\rho}_j^\top \overline{Z} - f_j\left(\overline{Z}\right) \geq \sup_{z \in \mathbb{Q}^N} \overline{\rho}_j^\top(z) - f_j(z) = f_j^{\#}(\overline{\rho}_j), \quad P\text{-a.s.}, \tag{19}$$

by continuity of f_j, which is the converse of (17), proving P-almost sure equality for $\rho = \overline{\rho}_j$ and thus (4). We next show that $\overline{\rho}_j$ satisfies the required integrability conditions, i.e., $\overline{\rho}_j \in L^{\infty-}(\mathbb{R}^N)$ and $f_j^{\#}(\overline{\rho}_j) \in L^{\infty-}(\mathbb{R})$. To this end, we first prove that $\overline{\rho}_j^\top Z \in L^{\infty-}(\mathbb{R})$ for any $Z \in L_j^{\infty-}(\mathbb{R}^N)$. Due to (18) and the Minkowski inequality and since $a \leq b$ implies $a_+ \leq |b|$, it follows for $Z \in L_j^{\infty-}(\mathbb{R}^N)$ that, for every $p \geq 1$,

$$\left(E\left[\left|\left(\overline{\rho}_j^\top Z\right)_+\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left|f_j\left(\overline{Z} + Z\right)\right|^p\right]\right)^{\frac{1}{p}} + \left(E\left[\left|f_j\left(\overline{Z}\right)\right|^p\right]\right)^{\frac{1}{p}} < \infty,$$

since f_j is of polynomial growth with 'random constant' $\alpha_j \in L_j^{\infty-}(\mathbb{R})$ and $\overline{Z}, Z$ are members of $L_j^{\infty-}(\mathbb{R}^N)$ by assumption. Applying the same argument to $\tilde{Z} = -Z$ yields

$$E\left[\left|\left(\overline{\rho}_j^\top Z\right)_-\right|^p\right] = E\left[\left|\left(\overline{\rho}_j^\top \tilde{Z}\right)_+\right|^p\right] < \infty,$$

since (18) holds for all $\mathcal{F}_j$-measurable random variables Z and $\tilde{Z}$ inherits the integrability of Z. We thus conclude that

$$E\left[\left|\overline{\rho}_j^\top Z\right|^p\right] < \infty \quad \text{and} \quad E\left[\left|\overline{\rho}_j\right|^p\right] < \infty,$$

where the second claim follows from the first by taking $Z = \operatorname{sgn}(\overline{\rho}_j)$ with the sign function applied componentwise. In order to show that $f_j^{\#}(\overline{\rho}_j) \in L^{\infty-}(\mathbb{R})$, we start with (4) and apply the Minkowski inequality to conclude that

$$\left(E\left[\left|f_j^{\#}\left(\overline{\rho}_j\right)\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left|\overline{\rho}_j^\top \overline{Z}\right|^p\right]\right)^{\frac{1}{p}} + \left(E\left[\left|f_j\left(\overline{Z}\right)\right|^p\right]\right)^{\frac{1}{p}} < \infty.$$

A.2 Proof of Theorem 3.3

We first show that the implications $(b) \Rightarrow (c) \Rightarrow (a)$ hold in an analogous formulation without the additional assumption that $G_i(z,y) = y$.

Proposition A.1. *Suppose (R) and (C), and consider the following assertions:*

(a) *The comparison principle (Comp) holds.*

(b') *For every $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ and $r \in \mathcal{A}_0^F$ the following positivity condition is fulfilled: For every $i = 0, \dots, J-1$*
$$(\rho_i^{(1)} + \rho_i^{(0)} r_i)^\top \beta_{i+1} \geq 0, \quad P\text{-a.s.}$$

(c') *For every $j = 0, \dots, J-1$ and any two random variables $Y^{(1)}, Y^{(2)} \in L^{\infty-}(\mathbb{R})$ with $Y^{(1)} \geq Y^{(2)}$ P-a.s., the monotonicity condition*
$$G_j(E_j[\beta_{j+1}Y^{(1)}], F_j(E_j[\beta_{j+1}Y^{(1)}])) \geq G_j(E_j[\beta_{j+1}Y^{(2)}], F_j(E_j[\beta_{j+1}Y^{(2)}])), \quad P\text{-a.s.},$$
is satisfied.

Then, $(b') \Rightarrow (c') \Rightarrow (a)$.

Proof. $(b') \Rightarrow (c')$: Fix $j \in \{0, \dots, J-1\}$ and let $Y^{(1)}$ and $Y^{(2)}$ be random variables which are in $L^{\infty-}(\mathbb{R})$ and satisfy $Y^{(1)} \geq Y^{(2)}$. By Lemma 2.2, there are $r \in \mathcal{A}_0^F$ and $(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G$ such that
$$F_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right]\right) = r_j^\top E_j\left[\beta_{j+1}Y^{(2)}\right] - F_j^{\#}(r_j)$$
and
$$\begin{aligned} &G_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right], F_j(E_j[\beta_{j+1}Y^{(1)}])\right) \\ &\qquad = \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(1)}\right] + \rho_j^{(0)} F_j\left(E_j[\beta_{j+1}Y^{(1)}]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right), \end{aligned}$$
P-almost surely. Hence, by (5), (b') and (6) we obtain
$$\begin{aligned} & G_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right], F_j\left(E_j\left[\beta_{j+1}Y^{(2)}\right]\right)\right) \\ \leq\ & \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(2)}\right] + \rho_j^{(0)} F_j\left(E_j[\beta_{j+1}Y^{(2)}]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\ =\ & E_j\left[\left(\rho_j^{(1)} + \rho_j^{(0)} r_j\right)^\top \beta_{j+1}Y^{(2)} - \rho_j^{(0)} F_j^{\#}(r_j) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\right] \\ \leq\ & E_j\left[\left(\rho_j^{(1)} + \rho_j^{(0)} r_j\right)^\top \beta_{j+1}Y^{(1)} - \rho_j^{(0)} F_j^{\#}(r_j) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\right] \\ \leq\ & \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y^{(1)}\right] + \rho_j^{(0)} F_j\left(E_j[\beta_{j+1}Y^{(1)}]\right) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\ =\ & G_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right], F_j\left(E_j\left[\beta_{j+1}Y^{(1)}\right]\right)\right). \end{aligned}$$

$(c') \Rightarrow (a)$: We prove this implication by backward induction. Let Y^{up} and Y^{low} respectively be super- and subsolutions of (2). Then the assertion is trivially true for $j = J$. Now assume, that the assertion is true for $j+1 \in \{1, \dots, J\}$. It follows by (c') and the definition of a sub- and supersolution that
$$Y_j^{up} \geq G_j(E_j[\beta_{j+1}Y_{j+1}^{up}], F_j(E_j[\beta_{j+1}Y_{j+1}^{up}])) \geq G_j(E_j[\beta_{j+1}Y_{j+1}^{low}], F_j(E_j[\beta_{j+1}Y_{j+1}^{low}])) \geq Y_j^{low}.$$
□

Proof of Theorem 3.3. Notice first, that under the additional assumption $G_i(z,y) = y$, assertions (b'), (c') in Proposition A.1 coincide with assertions (b), (c) in Theorem 3.3, because the vector $(0,...,0,1)^\top \in \mathbb{R}^{D+1}$ is the only control in $\mathcal{A}_0^G$ by linearity of G. It, hence, remains to show:

$(a) \Rightarrow (b)$: We prove the contraposition. Hence, we assume that there exists a $\overline{r} \in \mathcal{A}_0^F$ and a $j_0 \in \{0,\dots,J-1\}$ such that $P(\{\overline{r}_{j_0}^\top \beta_{j_0+1} < 0\}) > 0$. Then we define the process $\overline{Y}$ by

$$\overline{Y}_i = \begin{cases} Y_i, & i > j_0+1 \\ Y_{j_0+1} - n1_{\{\overline{r}_{j_0}^\top \beta_{j_0+1} < 0\}}, & i = j_0+1 \\ \overline{r}_i^\top E_i[\beta_{i+1}\overline{Y}_{i+1}] - F_i^\#(\overline{r}_i), & i \le j_0, \end{cases}$$

for $n \in \mathbb{N}$ which we fix later on. In view of (6), it follows easily that $\overline{Y}$ is a subsolution to (2). Now we observe that

$$\overline{Y}_{j_0} - Y_{j_0} = E_{j_0}\left[(\overline{r}_{j_0} - r_{j_0}^*)^\top \beta_{j_0+1} Y_{j_0+1}\right] + nE_{j_0}\left[(\overline{r}_{j_0}^\top \beta_{j_0+1})_-\right] - F_{j_0}^\#(\overline{r}_{j_0}) + F_{j_0}^\#(r_{j_0}^*),$$

where $r^* \in \mathcal{A}_{j_0}^F$ is such that for all $j = j_0,\dots,J-1$

$$(r_j^*)^\top E_j[\beta_{j+1}Y_{j+1}] - F_j^\#(r_j^*) = F_j(E_j[\beta_{j+1}Y_{j+1}]),$$

see Lemma 2.2. In a next step we define the set $A_{j_0,N}$ by

$$\begin{aligned} A_{j_0,N} \quad = \quad & \left\{E_{j_0}\left[(\overline{r}_{j_0}^\top \beta_{j_0+1})_-\right] \ge \frac{1}{N}\right\} \\ & \cap \left\{E_{j_0}\left[(\overline{r}_{j_0} - r_{j_0}^*)^\top \beta_{j_0+1} Y_{j_0+1}\right] - F_{j_0}^\#(\overline{r}_{j_0}) + F_{j_0}^\#(r_{j_0}^*) > -N\right\}. \end{aligned}$$

For $N \in \mathbb{N}$ sufficiently large (which is fixed from now on), we get that $P(A_{j_0,N}) > 0$ and therefore

$$(\overline{Y}_{j_0} - Y_{j_0})1_{A_{j_0,N}} > -N + \frac{n}{N} = 0$$

for $n = N^2$, which means that the comparison principle is violated for the subsolution $\overline{Y}$ (with this choice of n) and the (super-)solution Y.

□

A.3 Proof of Theorem 3.5

The proof of Theorem 3.5 is prepared by two propositions. The first of them shows almost sure optimality of optimal controls and martingales. This result completes, at the same time, the proof of Theorem 3.2.

Proposition A.2. *With the notation in Theorems 3.2 and 3.5, we have for every $i = j, \dots J$,*

$$\begin{aligned} Y_i &= \theta_i^{up}(\rho^{(1,*)}, \rho^{(0,*)}, r^*, M^*) = \Theta_i^{up}(\rho^{(1,*)}, \rho^{(0,*)}, M^*) \\ &= \theta_i^{low}(\rho^{(1,*)}, \rho^{(0,*)}, r^*, M^*) = \Theta_i^{low}(r^*, M^*). \end{aligned}$$

Proof. The proof is by backward induction on $i = J, \dots, j$. The case $i = J$ is obvious as all five processes have the same terminal condition ξ by construction. Now suppose the claim is already shown for $i+1 \in \{j+1,\dots,J\}$. Then, applying the induction hypothesis to the righthand side of the recursion formulas, we observe

$$\begin{aligned}\Theta_i^{up,*} &:= \Theta_i^{up}(\rho^{(1,*)},\rho^{(0,*)},M^*) = \theta_i^{up}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*),\\ \Theta_i^{low,*} &:= \Theta_i^{low}(\rho^{(1,*)},\rho^{(0,*)},M^*) = \theta_i^{low}(\rho^{(1,*)},\rho^{(0,*)},r^*,M^*).\end{aligned}$$

Exploiting again the induction hypothesis as well as the definition of the Doob martingale and the duality relation (8) we obtain

$$\begin{aligned}\Theta_i^{up,*} &= \left(\rho_i^{(1,*)}\right)^\top\left(\beta_{i+1}Y_{i+1}-\Delta M_{i+1}^*\right)+\rho_i^{(0,*)}F_i(\beta_{i+1}Y_{i+1}-\Delta M_{i+1}^*)-G_i^{\#}(\rho_i^{(1,*)},\rho_i^{(0,*)})\\ &= \left(\rho_i^{(1,*)}\right)^\top E_i[\beta_{i+1}Y_{i+1}]+\rho_i^{(0,*)}F_i(E_i[\beta_{i+1}Y_{i+1}])-G_i^{\#}(\rho_i^{(1,*)},\rho_i^{(0,*)})\\ &= G_i\left(E_i[\beta_{i+1}Y_{i+1}],F_i(E_i[\beta_{i+1}Y_{i+1}])\right)\\ &= Y_i.\end{aligned}$$

An analogous argument, making use of (9), shows $\Theta_i^{low,*}=Y_i$. □

The key step in the proof of Theorem 3.5 is the following alternative recursion formula for θ_j^{up} and θ_j^{low}. This result enables us to establish inequalities between Y_j, $E_j[\theta_j^{up}]$ and $E_j[\theta_j^{low}]$, replacing, in a sense, the comparison principle (Comp).

Proposition A.3. *Suppose (R) and (C) and let $M\in\mathcal{M}_D$. Then, for every $j=0,\dots,J$ and $\left(\rho^{(1)},\rho^{(0)}\right)\in\mathcal{A}_j^G$ and $r\in\mathcal{A}_j^F$, we have for all $i=j,\dots,J$ the P-almost sure identities*

$$\begin{aligned}&\theta_i^{up}(\rho^{(1)},\rho^{(0)},r,M)\\ &\qquad= \sup_{u\in\mathbb{R}^D}\Phi_{i+1}\left(\rho_i^{(1)},\rho_i^{(0)},u,\theta_{i+1}^{up}(\rho^{(1)},\rho^{(0)},r,M),\theta_{i+1}^{low}(\rho^{(1)},\rho^{(0)},r,M),\Delta M_{i+1}\right)\end{aligned}$$

$$\begin{aligned}&\theta_i^{low}(\rho^{(1)},\rho^{(0)},r,M)\\ &\qquad= \inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}}\Phi_{i+1}\left(v^{(1)},v^{(0)},r_i,\theta_{i+1}^{low}(\rho^{(1)},\rho^{(0)},r,M),\theta_{i+1}^{up}(\rho^{(1)},\rho^{(0)},r,M),\Delta M_{i+1}\right),\end{aligned}$$

where $\Phi_{J+1}(v^{(1)},v^{(0)},u,\vartheta_1,\vartheta_2,m)=\xi$ and

$$\begin{aligned}&\Phi_{i+1}\left(v^{(1)},v^{(0)},u,\vartheta_1,\vartheta_2,m\right)\\ &= \left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_+\vartheta_1-\left(\left(v^{(1)}\right)^\top\beta_{i+1}\right)_-\vartheta_2-\left(v^{(1)}\right)^\top m\\ &\quad+v^{(0)}\left(\left(u^\top\beta_{i+1}\right)_+\vartheta_1-\left(u^\top\beta_{i+1}\right)_-\vartheta_2-u^\top m-F_i^{\#}(u)\right)-G_i^{\#}\left(v^{(1)},v^{(0)}\right)\end{aligned}$$

for $i=j,...,J-1$. In particular, $\theta_i^{low}(\rho^{(1)},\rho^{(0)},r,M)\le\theta_i^{up}(\rho^{(1)},\rho^{(0)},r,M)$ for every $i=j,\dots,J$.

Proof. First we fix $j\in\{0,\dots,J-1\}$, $M\in\mathcal{M}_D$ and controls $\left(\rho^{(1)},\rho^{(0)}\right)$ and r in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$ and define θ^{up} and θ^{low} by (10). To lighten the notation, we set

$$\Phi_{i+1}^{low}\left(v^{(1)},v^{(0)},r_i\right)=\Phi_{i+1}\left(v^{(1)},v^{(0)},r_i,\theta_{i+1}^{low}(\rho^{(1)},\rho^{(0)},r,M),\theta_{i+1}^{up}(\rho^{(1)},\rho^{(0)},r,M),\Delta M_{i+1}\right)$$

and define Φ^{up}_{i+1} accordingly (interchanging the roles of θ^{up} and θ^{low}). We show the assertion by backward induction on $i = J, \ldots, j$ with the case $i = J$ being trivial since $\theta^{up}_J = \theta^{low}_J = \Phi_{J+1} = \xi$ by definition. Now suppose that the assertion is true for $i+1$. For any $(v^{(1)}, v^{(0)}) \in \mathbb{R}^{D+1}$ we obtain, by (5), the following upper bound for θ^{low}_i:

$$\begin{aligned}
&\Phi^{low}_{i+1}\left(v^{(1)}, v^{(0)}, r_i\right) \\
&= \left(v^{(1)}\right)^\top \left(\beta_{i+1}\left(\theta^{low}_{i+1} 1_{\{(v^{(1)})^\top \beta_{i+1} \geq 0\}} + \theta^{up}_{i+1} 1_{\{(v^{(1)})^\top \beta_{i+1} < 0\}}\right) - \Delta M_{i+1}\right) \\
&\quad + v^{(0)}\left(\left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right) \\
&\geq G_i\Bigg(\beta_{i+1}\left(\theta^{low}_{i+1} 1_{\{(v^{(1)})^\top \beta_{i+1} \geq 0\}} + \theta^{up}_{i+1} 1_{\{(v^{(1)})^\top \beta_{i+1} < 0\}}\right) - \Delta M_{i+1}, \\
&\qquad \left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) \\
&\geq \min_{\iota \in \{up, low\}} G_i\Bigg(\beta_{i+1}\theta^{\iota}_{i+1} - \Delta M_{i+1}, \left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} \\
&\qquad - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) \\
&= \theta^{low}_i.
\end{aligned}$$

We emphasize that this chain of inequalities holds for every $\omega \in \Omega$, and not only P-almost surely. Hence,

$$\inf_{(v^{(1)}, v^{(0)}) \in \mathbb{R}^{D+1}} \Phi^{low}_{i+1}\left(v^{(1)}, v^{(0)}, r_i\right) \geq \theta^{low}_i$$

for every $\omega \in \Omega$. To conclude the argument for θ^{low}_i, it remains to show that the converse inequality holds P-almost surely. Thanks to (5) we get

$$\begin{aligned}
&G_i\left(\beta_{i+1}\theta^{\iota}_{i+1} - \Delta M_{i+1}, \left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) \\
&= \inf_{\left(v^{(1)}, v^{(0)}\right) \in \mathbb{R}^{D+1}} \left(v^{(1)}\right)^\top \left(\beta_{i+1}\theta^{\iota}_{i+1} - \Delta M_{i+1}\right) + v^{(0)}\Bigg(\left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} \\
&\qquad - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right).
\end{aligned}$$

Together with $\theta^{up}_{i+1} \geq \theta^{low}_{i+1}$ P-a.s. (by the induction hypothesis) we obtain

$$\begin{aligned}
\theta^{low}_i &= \min_{\iota \in \{up, low\}} G_i\Bigg(\beta_{i+1}\theta^{\iota}_{i+1} - \Delta M_{i+1}, \left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} \\
&\quad - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\Bigg) \\
&= \min_{\iota \in \{up, low\}} \inf_{\left(v^{(1)}, v^{(0)}\right) \in \mathbb{R}^{D+1}} \left(v^{(1)}\right)^\top \beta_{i+1}\theta^{\iota}_{i+1} - \left(v^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + v^{(0)}\left(\left(r_i^\top \beta_{i+1}\right)_+ \theta^{low}_{i+1} - \left(r_i^\top \beta_{i+1}\right)_- \theta^{up}_{i+1} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}} \left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{up} - \left(v^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + v^{(0)}\left(\left(r_i^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(r_i^\top \beta_{i+1}\right)_- \theta_{i+1}^{up} - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right) \\
&= \inf_{\left(v^{(1)},v^{(0)}\right)\in\mathbb{R}^{D+1}} \Phi_{i+1}^{low}\left(v^{(1)}, v^{(0)}, r_i\right), \quad P\text{-a.s.}
\end{aligned}$$

We next turn to θ_i^{up} where the overall strategy of proof is similar. Recall first that the monotonicity of G in the y-component implies existence of a set $\bar{\Omega}_\rho$ (depending on $\rho^{(0)}$) of full P-measure such that $\rho_k^{(0)}(\omega) \geq 0$ for every $\omega \in \bar{\Omega}_\rho$ and $k = j, \ldots, J-1$. By (6) we find that, for any $u \in \mathbb{R}^D$, $\Phi_{i+1}^{up}(\rho_i^{(0)}, \rho_i^{(1)}, u)$ is a lower bound for θ_i^{up} on $\bar{\Omega}_\rho$:

$$\begin{aligned}
&\Phi_{i+1}^{up}\left(\rho_i^{(1)}, \rho_i^{(0)}, u\right) \\
&= \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)}\left(\left(u^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(u^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&\leq \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} F_i\left(\beta_{i+1}\left(\theta_{i+1}^{up} 1_{\{u^\top \beta_{i+1} \geq 0\}} + \theta_{i+1}^{low} 1_{\{u^\top \beta_{i+1} < 0\}}\right) - \Delta M_{i+1}\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&\leq \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \max_{\iota\in\{up,low\}} F_i(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&= \theta_i^{up}.
\end{aligned}$$

Hence,

$$\sup_{u\in\mathbb{R}^D} \Phi_{i+1}^{up}\left(\rho_i^{(1)}, \rho_i^{(0)}, u\right) \leq \theta_i^{up}$$

on $\bar{\Omega}_\rho$, and, thus, P-almost surely. To complete the proof of the proposition, we show the converse inequality. As $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ P-a.s., we conclude, by (6),

$$\begin{aligned}
\theta_i^{up} &= \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \max_{\iota\in\{up,low\}} F_i(\beta_{i+1}\theta_{i+1}^{\iota} - \Delta M_{i+1}) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&= \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad + \rho_i^{(0)} \max_{\iota\in\{up,low\}} \sup_{u\in\mathbb{R}^D} \left(u^\top \beta_{i+1}\theta_{i+1}^{\iota} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) \\
&\quad - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&\leq \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1}
\end{aligned}$$

$$
\begin{aligned}
&+\rho_i^{(0)} \sup_{u\in\mathbb{R}^D} \left(\left(u^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(u^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) \\
&-G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
= \;& \sup_{u\in\mathbb{R}^D} \Phi_{i+1}^{up}\left(\rho_i^{(1)}, \rho_i^{(0)}, u\right), \quad P\text{-a.s.}
\end{aligned}
$$

As $\Phi_{i+1}\left(v^{(1)}, v^{(0)}, u, \vartheta_1, \vartheta_2, m\right)$ is increasing in ϑ_1 and decreasing in ϑ_2, we finally get

$$
\begin{aligned}
\theta_i^{up} &= \sup_{u\in\mathbb{R}^D} \Phi_{i+1}\left(\rho_i^{(1)}, \rho_i^{(0)}, u, \theta_{i+1}^{up}, \theta_{i+1}^{low}, \Delta M_{i+1}\right) \geq \sup_{u\in\mathbb{R}^D} \Phi_{i+1}\left(\rho_i^{(1)}, \rho_i^{(0)}, u, \theta_{i+1}^{low}, \theta_{i+1}^{up}, \Delta M_{i+1}\right) \\
&\geq \inf_{\left(v^{(1)}, v^{(0)}\right)\in\mathbb{R}^{D+1}} \Phi_{i+1}\left(v^{(1)}, v^{(0)}, r_i, \theta_{i+1}^{low}, \theta_{i+1}^{up}, \Delta M_{i+1}\right) = \theta_i^{low}, \quad P\text{-a.s.},
\end{aligned}
$$

as $\theta_{i+1}^{up} \geq \theta_{i+1}^{low}$ P-a.s. by the induction hypothesis. □

We are now in the position to complete the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $j \in \{0, \ldots, J-1\}$ be fixed from now on. Due to Proposition A.2 it only remains to show that

$$E_i[\theta_i^{low}] \leq Y_i \leq E_i[\theta_i^{up}]$$

for $i = j, ..., J$. We prove this by backward induction on i. Therefore, we fix $M \in \mathcal{M}_D$ and controls $\left(\rho^{(1)}, \rho^{(0)}\right)$ and r in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$, as well as 'optimizers' $\left(\rho^{(1,*)}, \rho^{(0,*)}\right)$ and r^* in $\mathcal{A}_j^G$ respectively $\mathcal{A}_j^F$ which satisfy the duality relations (8) and (9). By definition of θ^{up} and θ^{low} the assertion is trivially true for $i = J$. Suppose that the assertion is true for $i+1$. Recalling Proposition A.3 and applying the tower property of the conditional expectation as well as the induction hypothesis, we get

$$
\begin{aligned}
E_i[\theta_i^{low}] &= E_i\Bigg[\inf_{\left(v^{(1)}, v^{(0)}\right)\in\mathbb{R}^{D+1}} \left(\left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(\left(v^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{up}\right. \\
&\quad - \left(v^{(1)}\right)^\top \Delta M_{i+1} + v^{(0)}\left(\left(r_i^\top \beta_{i+1}\right)_+ \theta_{i+1}^{low} - \left(r_i^\top \beta_{i+1}\right)_- \theta_{i+1}^{up} - r_i^\top \Delta M_{i+1}\right. \\
&\quad \left.\left. - F_i^{\#}(r_i)\right) - G_i^{\#}\left(v^{(1)}, v^{(0)}\right)\right] \\
&\leq E_i\Bigg[\left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{up}\right] \\
&\quad - \left(\rho_i^{(1,*)}\right)^\top \Delta M_{i+1} + \rho_i^{(0,*)}\left(\left(r_i^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(r_i^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{up}\right]\right. \\
&\quad \left. - r_i^\top \Delta M_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right)\Bigg] \\
&\leq E_i\left[\left(\rho_i^{(1,*)}\right)^\top \beta_{i+1} Y_{i+1} + \rho_i^{(0,*)}\left(r_i^\top \beta_{i+1} Y_{i+1} - F_i^{\#}(r_i)\right) - G_i^{\#}\left(\rho_i^{(1,*)}, \rho_i^{(0,*)}\right)\right] \\
&\leq G_i\left(E_i\left[\beta_{i+1} Y_{i+1}\right], F_i\left(E_i\left[\beta_{i+1} Y_{i+1}\right]\right)\right) \\
&= Y_i.
\end{aligned}
$$

Here, the last inequality is an immediate consequence of (6), the nonnegativity of $\rho_i^{(0,*)}$ and the duality relation (8). Applying an analogous argument, we obtain that $E_i[\theta_i^{up}] \geq Y_i$. Indeed,

$$\begin{aligned}
E_i[\theta_i^{up}] &= E_i\Bigg[\left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad +\rho_i^{(0)} \sup_{u\in\mathbb{R}^D} \left(\left(u^\top \beta_{i+1}\right)_+ \theta_{i+1}^{up} - \left(u^\top \beta_{i+1}\right)_- \theta_{i+1}^{low} - u^\top \Delta M_{i+1} - F_i^{\#}(u)\right) \\
&\quad -G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right)\Bigg] \\
&\geq E_i\Bigg[\left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{up}\right] - \left(\left(\rho_i^{(1)}\right)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{low}\right] - \left(\rho_i^{(1)}\right)^\top \Delta M_{i+1} \\
&\quad +\rho_i^{(0)}\Big(\left((r_i^*)^\top \beta_{i+1}\right)_+ E_{i+1}\left[\theta_{i+1}^{up}\right] - \left((r_i^*)^\top \beta_{i+1}\right)_- E_{i+1}\left[\theta_{i+1}^{low}\right] - (r_i^*)^\top \Delta M_{i+1} \\
&\quad -F_i^{\#}(r_i^*)\Big) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right)\Bigg] \\
&\geq \left(\rho_i^{(1)}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] + \rho_i^{(0)}\left((r_i^*)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] - F_i^{\#}(r_i^*)\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&= \left(\rho_i^{(1)}\right)^\top E_i\left[\beta_{i+1}Y_{i+1}\right] + \rho_i^{(0)} F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right) - G_i^{\#}\left(\rho_i^{(1)}, \rho_i^{(0)}\right) \\
&\geq G_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right], F_i\left(E_i\left[\beta_{i+1}Y_{i+1}\right]\right)\right) \\
&= Y_i.
\end{aligned}$$

making now use of the nonnegativity of $\rho_i^{(0)}$, the duality relation (9), and (5). This establishes

$$E_i[\theta_i^{low}] \leq Y_i \leq E_i[\theta_i^{up}]$$

for $i = j, \ldots, J$. □

A.4 Proof of Proposition 3.6

By the definition of θ^{up}, the tower property of the conditional expectation, Jensen's inequality (applied to the convex functions max and F_j), and the comparison $Y_{j+1}^{up} \geq Y_{j+1}^{low}$ (in view of Proposition A.3), we obtain

$$\begin{aligned}
Y_j^{up} &= E_j\Bigg[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ \theta_{j+1}^{up} - \left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- \theta_{j+1}^{low} - \left(\rho_j^{(1)}\right)^\top \Delta M_{j+1} \\
&\quad + \rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j(\beta_{j+1}\theta_{j+1}^{\iota} - \Delta M_{j+1}) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)\Bigg] \\
&\geq E_j\left[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_+ Y_{j+1}^{up}\right] - E_j\left[\left(\left(\rho_j^{(1)}\right)^\top \beta_{j+1}\right)_- Y_{j+1}^{low}\right] \\
&\quad +\rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j(E_j[\beta_{j+1}Y_{j+1}^{\iota}]) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{up}\right] + \rho_j^{(0)} \max_{\iota\in\{up,low\}} F_j(E_j[\beta_{j+1}Y_{j+1}^{\iota}]) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right) \\
&\geq \left(\rho_j^{(1)}\right)^\top E_j\left[\beta_{j+1}Y_{j+1}^{up}\right] + \rho_j^{(0)} F_j(E_j[\beta_{j+1}Y_{j+1}^{up}]) - G_j^{\#}\left(\rho_j^{(1)}, \rho_j^{(0)}\right)
\end{aligned}$$

$$\geq \quad G_j(E_j[\beta_{j+1}Y^{up}_{j+1}], F_j(E_j[\beta_{j+1}Y^{up}_{j+1}])),$$

making, again, use of the nonnegativity of $\rho^{(0)}_j$ and (5). As in the previous proofs, the argument for the lower bound Y^{low} is essentially the same, and we skip the details.

A.5 Proofs for Section 4

We first derive the alternative representation for $\Theta^{low}_0(r, M)$ as a pathwise minimization problem. To this end, define, for some fixed control $r \in \mathcal{A}^F_0$ and martingale $M \in \mathcal{M}_D$,

$$\begin{aligned}\tilde{\Theta}^{low}_i \quad = \quad & \inf_{(v^{(1)}_j, v^{(0)}_j)\in\mathbb{R}^{D+1},\ j=i,\dots,J-1} \Bigg(w_{i,J}(v^{(1)}, v^{(0)}, r)\xi \\ & - \sum_{j=i}^{J-1} w_{i,j}(v^{(1)}, v^{(0)}, r) \left(v^{(0)}_j F^{\#}_j(r_j) + (v^{(1)}_j + v^{(0)}_j r_j)^\top \Delta M_{j+1} + G^{\#}_j(v^{(1)}_j, v^{(0)}_j) \right) \Bigg),\end{aligned}$$

where

$$w_{i,j}(v^{(1)}, v^{(0)}, u) = \prod_{k=i}^{j-1} (v^{(1)}_k + v^{(0)}_k u_k)^\top \beta_{k+1}.$$

Then, $\tilde{\Theta}^{low}_J = \xi$ and, for $i = 0, \dots, J-1$,

$$\begin{aligned}\tilde{\Theta}^{low}_i \quad = \quad & \inf_{(v^{(1)}_i, v^{(0)}_i)\in\mathbb{R}^{D+1}} \ \inf_{(v^{(1)}_j, v^{(0)}_j)\in\mathbb{R}^{D+1},\ j=i+1,\dots,J-1} \Bigg((v^{(1)}_i + v^{(0)}_i r_i)^\top \beta_{i+1} \Bigg(w_{i+1,J}(v^{(1)}, v^{(0)}, r)\xi \\ & - \sum_{j=i+1}^{J-1} w_{i+1,j}(v^{(1)}, v^{(0)}, r) \left(v^{(0)}_j F^{\#}_j(r_j) + (v^{(1)}_j + v^{(0)}_j r_j)^\top \Delta M_{j+1} + G^{\#}_j(v^{(1)}_j, v^{(0)}_j) \right) \Bigg) \\ & - \left(v^{(0)}_i F^{\#}_i(r_i) + (v^{(1)}_i + v^{(0)}_i r_i)^\top \Delta M_{i+1} + G^{\#}_i(v^{(1)}_i, v^{(0)}_i) \right) \Bigg).\end{aligned}$$

The outer infimum can be taken restricted to such $(v^{(1)}_i, v^{(0)}_i) \in \mathbb{R}^{D+1}$ which belong to $D_{G^{\#}_i(\omega,\cdot)}$, because the expression which is to be minimized is $+\infty$ otherwise. Then, assumption (12) implies that the inner infimum can be interchanged with the nonnegative factor $(v^{(1)}_i + v^{(0)}_i r_i)^\top \beta_{i+1}$, which yields

$$\begin{aligned}\tilde{\Theta}^{low}_i \quad = \quad & \inf_{(v^{(1)}_i, v^{(0)}_i)\in D_{G^{\#}_i(\omega,\cdot)}} \Big((v^{(1)}_i + v^{(0)}_i r_i)^\top \beta_{i+1} \tilde{\Theta}^{low}_{i+1} \\ & - \left(v^{(0)}_i F^{\#}_i(r_i) + (v^{(1)}_i + v^{(0)}_i r_i)^\top \Delta M_{i+1} + G^{\#}_j(v^{(1)}_i, v^{(0)}_i) \right) \Big),\end{aligned}$$

and the infimum can, by the same argument as above, again be replaced by that over the whole $\mathbb{R}^{D+1}$. Following the lines of the proof of Proposition A.3 with the simpler Θ^{low} instead of θ^{low} we observe that the very same recursion formula is satisfied P-almost surely by $\Theta^{low}_i(r, M)$. In particular,

$$\Theta^{low}_0(r, M) \quad = \quad \tilde{\Theta}^{low}_0 = \inf_{(v^{(1)}_j, v^{(0)}_j)\in\mathbb{R}^{D+1},\ j=0,\dots,J-1} \Bigg(w_J(v^{(1)}, v^{(0)}, r)\xi$$

$$-\sum_{j=0}^{J-1} w_j(v^{(1)}, v^{(0)}, r) \left(v_j^{(0)} F_j^{\#}(r_j) + (v_j^{(1)} + v_j^{(0)} r_j)^\top \Delta M_{j+1} + G_j^{\#}(v_j^{(1)}, v_j^{(0)}) \right) \Bigg),$$

P-almost surely. The alternative expression for $\Theta_0^{up}(\rho^{(1)}, \rho^{(0)}, M)$ as a pathwise maximization problem can be shown in the same way. It, thus, remains to show that Y_0 is the equilibrium value of the two-player game. We apply the alternative representations for Θ^{up} and Θ^{low} as pathwise optimization problems as well as Theorem 3.2 (twice) in order to conclude

$$\begin{aligned}
Y_0 &= \sup_{r \in \mathcal{A}_0^F,\, M \in \mathcal{M}_D} E[\Theta_0^{low}(r, M)] \\
&= \sup_{r \in \mathcal{A}_0^F,\, M \in \mathcal{M}_D} E\Bigg[\inf_{(v_j^{(1)}, v_j^{(0)}) \in \mathbb{R}^{D+1},\, j=0,\dots,J-1} \Bigg(w_J(v^{(1)}, v^{(0)}, r)\xi \\
&\quad - \sum_{j=0}^{J-1} w_j(v^{(1)}, v^{(0)}, r) \left(v_j^{(0)} F_j^{\#}(r_j) + (v_j^{(1)} + v_j^{(0)} r_j)^\top \Delta M_{j+1} + G_j^{\#}(v_j^{(1)}, v_j^{(0)}) \right) \Bigg) \Bigg] \\
&\le \sup_{r \in \mathcal{A}_0^F,\, M \in \mathcal{M}_D} \inf_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G} E\Bigg[w_J(\rho^{(1)}, \rho^{(0)}, r)\xi \\
&\quad - \sum_{j=0}^{J-1} w_j(\rho^{(1)}, \rho^{(0)}, r) \left(\rho_j^{(0)} F_j^{\#}(r_j) + (\rho_j^{(1)} + \rho_j^{(0)} r_j)^\top \Delta M_{j+1} + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}) \right) \Bigg] \\
&= \sup_{r \in \mathcal{A}_0^F} \inf_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G} E\Bigg[w_J(\rho^{(1)}, \rho^{(0)}, r)\xi - \sum_{j=0}^{J-1} w_j(\rho^{(1)}, \rho^{(0)}, r) \left(\rho_j^{(0)} F_j^{\#}(r_j) + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}) \right) \Bigg] \\
&\le \inf_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G} \sup_{r \in \mathcal{A}_0^F} E\Bigg[w_J(\rho^{(1)}, \rho^{(0)}, r)\xi - \sum_{j=0}^{J-1} w_j(\rho^{(1)}, \rho^{(0)}, r) \left(\rho_j^{(0)} F_j^{\#}(r_j) + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}) \right) \Bigg] \\
&\le \inf_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G,\, M \in \mathcal{M}_D} E\Bigg[\sup_{(u_j) \in \mathbb{R}^D,\, j=0,\dots,J-1} \Bigg(w_J(\rho^{(1)}, \rho^{(0)}, u)\xi \\
&\quad - \sum_{j=0}^{J-1} w_j(\rho^{(1)}, \rho^{(0)}, u) \left(\rho_j^{(0)} F_j^{\#}(u_j) + (\rho_j^{(1)} + \rho_j^{(0)} u_j)^\top \Delta M_{j+1} + G_j^{\#}(\rho_j^{(1)}, \rho_j^{(0)}) \right) \Bigg) \Bigg] \\
&= \inf_{(\rho^{(1)}, \rho^{(0)}) \in \mathcal{A}_0^G,\, M \in \mathcal{M}_D} E[\Theta_0^{up}(\rho^{(1)}, \rho^{(0)}, M)] = Y_0.
\end{aligned}$$

Here we also applied the zero-expectation property of our penalties at adapted controls. Consequently, all inequalities turn into equalities, which completes the proof.

References

S. Alanko and M. Avellaneda. Reducing variance in the numerical solution of BSDEs. *Comptes Rendus Mathematique*, 351(3):135–138, 2013.

L. Andersen and M. Broadie. Primal-dual simulation algorithm for pricing multidimensional American options. *Management Science*, 50(9):1222–1234, 2004.

M. Avellaneda, A. Levy, and A. Parás. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, 2(2):73–88, 1995.

C. Bender and J. Steiner. Least-squares Monte Carlo for Backward SDEs. In R. Carmona, P. Del Moral, P. Hu, and N. Oudjane, editors, *Numerical Methods in Finance*, pages 257–289. Springer, 2012.

C. Bender, N. Schweizer, and J. Zhuo. A primal-dual algorithm for BSDEs. *Mathematical Finance*, Early View, DOI: 10.1111/mafi.12100, 2015.

D. P. Bertsekas. *Dynamic programming and optimal control*, volume I. Athena Scientific, 3rd edition, 2005.

C. Beveridge and M. Joshi. Monte Carlo bounds for game options including convertible bonds. *Management Science*, 57(5):960–974, 2011.

D. Brigo and F. Mercurio. *Interest Rate Models – Theory and Practice: With Smile, Inflation and Credit*. Springer, 2nd edition, 2006.

D. Brigo and A. Pallavicini. Counterparty risk pricing under correlation between default and interest rates. In J. H. Miller, D. C. Edelman, and J. A. D. Appleby, editors, *Numerical Methods for Finance*, pages 63–81. Chapman and Hall/CRC, 2007.

D. Brigo, M. Morini, and A. Pallavicini. *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. John Wiley & Sons, 2013.

D. B. Brown and J. E. Smith. Dynamic portfolio optimization with transaction costs: Heuristics and dual bounds. *Management Science*, 57(10):1752–1770, 2011.

D. B. Brown, J. E. Smith, and P. Sun. Information relaxations and duality in stochastic dynamic programs. *Operations Research*, 58(4):785–801, 2010.

P. Cheridito, M. Kupper, and N. Vogelpoth. Conditional analysis on $\mathbb{R}^d$. In A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff, and C. Schrage, editors, *Set Optimization and Applications in Finance - The State of the Art*. Springer, forthcoming, 2015.

S. Crépey. *Financial Modeling*. Springer, 2013.

S. Crépey, R. Gerboud, Z. Grbac, and N. Ngor. Counterparty risk and funding: The four wings of the TVA. *International Journal of Theoretical and Applied Finance*, 16(2):1350006, 2013.

A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. *Annals of Applied Probability*, 21(4):1322–1364, 2011.

P. Glasserman and B. Yu. Simulation for American options: Regression now or regression later? In H. Niederreiter, editor, *Monte Carlo and Quasi-Monte Carlo Methods 2002*, pages 213–226. Springer, 2004.

J. Guyon and P. Henry-Labordère. The uncertain volatility model: A Monte Carlo approach. *Journal of Computational Finance*, 14(3):37–71, 2011.

J. Guyon and P. Henry-Labordère. *Nonlinear Option Pricing*. Chapman and Hall/CRC, 2013.

M. Haugh and A. E. B. Lim. Linear–quadratic control and information relaxations. *Operations Research Letters*, 40(6):521–528, 2012.

M. B. Haugh and L. Kogan. Pricing American options: A duality approach. *Operations Research*, 52(2):258–270, 2004.

M. B. Haugh and C. Wang. Information relaxations and dynamic zero-sum games. *arXiv preprint 1405.4347*, 2015.

H. Kraft and F. T. Seifried. Stochastic differential utility as the continuous-time limit of recursive utility. *Journal of Economic Theory*, 151:528–550, 2014.

C. Kühn, A. E. Kyprianou, and K. van Schaik. Pricing Israeli options: A pathwise approach. *Stochastics*, 79(1-2):117–137, 2007.

R. Lord, R. Koekkoek, and D. van Dijk. A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance*, 10(2):177–194, 2010.

T. J. Lyons. Uncertain volatility and the risk-free synthesis of derivatives. *Applied Mathematical Finance*, 2(2):117–133, 1995.

H. Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer, 2009.

W. B. Powell. *Approximate Dynamic Programming: Solving the curses of dimensionality*. John Wiley & Sons, 2nd edition, 2011.

N. S. Rasmussen. Control variates for Monte Carlo valuation of American options. *Journal of Computational Finance*, 9(1):84–102, 2005.

R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

L. C. G. Rogers. Monte Carlo valuation of American options. *Mathematical Finance*, 12(3): 271–286, 2002.

L. C. G. Rogers. Pathwise stochastic optimal control. *SIAM Journal on Control and Optimization*, 46(3):1116–1132, 2007.

J. M. Vanden. Exact superreplication strategies for a class of derivative assets. *Applied Mathematical Finance*, 13(01):61–87, 2006.

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[8] L. Denis, D. A. Lorenz, and D. Trede. Greedy Solution of Ill-Posed Problems: Error Bounds and Exact Inversion. Preprint 8, DFG-SPP 1324, April 2009.

[9] U. Friedrich. A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs. Preprint 9, DFG-SPP 1324, April 2009.

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[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.

[23] F.J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter. Multi-level Monte Carlo Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 23, DFG-SPP 1324, August 2009.

[24] S. Dereich and F. Heidenreich. A Multilevel Monte Carlo Algorithm for Lévy Driven Stochastic Differential Equations. Preprint 24, DFG-SPP 1324, August 2009.

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[40] B. Niu, F.J. Hickernell, T. Müller-Gronbach, and K. Ritter. Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 40, DFG-SPP 1324, March 2010.

[41] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Construction of Compactly Supported Shearlet Frames. Preprint 41, DFG-SPP 1324, March 2010.

[42] C. Bender and J. Steiner. Error Criteria for Numerical Solutions of Backward SDEs. Preprint 42, DFG-SPP 1324, April 2010.

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[48] S. Kestler and K. Urban. Adaptive Wavelet Methods on Unbounded Domains. Preprint 48, DFG-SPP 1324, June 2010.

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[65] W. Hackbusch. Tensorisation of Vectors and their Efficient Convolution. Preprint 65, DFG-SPP 1324, November 2010.

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[67] E. Novak and H. Woźniakowski. On the Power of Function Values for the Approximation Problem in Various Settings. Preprint 67, DFG-SPP 1324, November 2010.

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[78] G. Plonka, S. Tenorth, and A. Iske. Optimal Representation of Piecewise Hölder Smooth Bivariate Functions by the Easy Path Wavelet Transform. Preprint 78, DFG-SPP 1324, January 2011.

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[90] P. Binev, F. Blanco-Silva, D. Blom, W. Dahmen, P. Lamby, R. Sharpley, and T. Vogt. High Quality Image Formation by Nonlocal Means Applied to High-Angle Annular Dark Field Scanning Transmission Electron Microscopy (HAADF-STEM). Preprint 90, DFG-SPP 1324, March 2011.

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[94] H.-C. Kreusler and H. Yserentant. The Mixed Regularity of Electronic Wave Functions in Fractional Order and Weighted Sobolev Spaces. Preprint 94, DFG-SPP 1324, June 2011.

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[124] P.A. Cioica, S. Dahlke, N. Döhring, U. Friedrich, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R.L. Schilling. On the convergence analysis of Rothe's method. Preprint 124, DFG-SPP 1324, July 2012.

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[167] T. Müller-Gronbach and L. Yaroslavtseva. Deterministic quadrature formulas for SDEs based on simplified weak Ito-Taylor steps. Preprint 167, DFG-SPP 1324, June 2014.

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[169] S. Dahlke, N. Döhring, and S. Kinzel. A class of random functions in non-standard smoothness spaces. Preprint 169, DFG-SPP 1324, September 2014.

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[174] P.A. Cioica, S. Dahlke, N. Dhring, U. Friedrich, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R.L. Schilling. On the convergence analysis of the inexact linearly implicit Euler scheme for a class of SPDEs. Preprint 174, DFG-SPP 1324, January 2015.

[175] C. Bender, C. Gärtner, and N. Schweizer. Pathwise Dynamic Programming. Preprint 175, DFG-SPP 1324, August 2015.