Domain Decomposition Strategies for the Stochastic Heat Equation

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Contents

Problem setting

- Stochastic heat equation on an open bounded polygonal domain $\mathcal{D} \subset \mathbb{R}^d$

$$\begin{aligned} \mathrm{d}X(t) - \Delta X(t)\mathrm{d}t &= Q^{1/2}\mathrm{d}W(t), \quad 0 < t < T \\ X(0) &= X_0 \end{aligned}$$

- $X_0 \in L^2(\Omega; L^2(\mathcal{D}))$, W cylindrical Wiener process on $L^2(\mathcal{D})$
- Q linear, nonnegative, symmetric, bounded from $L^2(\mathcal{D})$ into $D(\Delta^\beta)$

Then we have:

- Existence and uniqueness of mild solutions
- Regularity: $\sup_{t\in[0,T]} \left(\mathbb{E}\|X(t)\|^2_{H^1}\right)^{1/2} \leq C(T, \|X_0\|_{H^1})$

Backward Euler scheme: Known results

- $N > 0$ temporal mesh, $\tau = T/N$, $h > 0$ spatial mesh
- Continuous finite elements $S_0^h(\mathcal{D})$
- $Y_h^n \in L^2(\Omega; S_0^h(\mathcal{D}))$ solves

$$\begin{aligned} (Y_h^n - Y_h^{n-1}, v_h) + \tau(\nabla Y_h^n, \nabla v_h) &= (\sqrt{\tau} Q^{1/2} \chi^{n-1}, v_h) \\ (Y_h^0, v_h) &= (X_0, v_h), \end{aligned}$$

  for all $v_h \in S_0^h(\mathcal{D})$.
- Existence and uniqueness of discrete solution $\{Y_h^n\}$
- Stability: Energy estimate
- For $\mathrm{Tr}(\Delta Q) < \infty$

$$\max_{0 \le n \le N} \mathbb{E}\|Y_h^n - Y_h^{n-1}\|_{L^2}^2 \le C\tau$$

Convergence properties of backward Euler

- Strong convergence (Yan, 2005)

$$\max_{0\leq n\leq N}\left(\mathbb{E}\|Y_h^n - X(t_n)\|_{L^2}^2\right)^{1/2} \leq C(\tau^{1/2}+h)$$

- Weak convergence (Debussche, Printems, 2007)

$$\max_{0\leq n\leq N}\left|\mathbb{E}\left[\phi(Y_h^n) - \phi(X(t_n))\right]\right| \leq C(\tau^{\gamma}+h^{2\gamma})$$

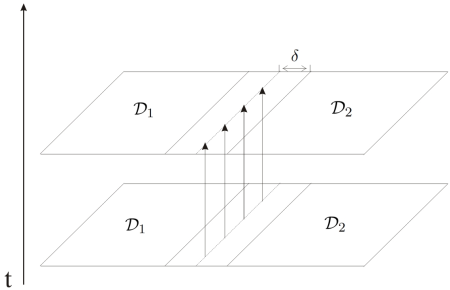
for $0<\gamma<1-d/2+\beta$, where $Q\in L(L^2(\mathcal{D}), D(\Delta^{\beta}))$.

Solution of linear system of dimension $O(h^{-d})$
STRATEGY: domain decomposition methods

Problem setting for Scheme A

- (Blum, Lisky, Rannacher, 1992) Consider

$$\partial_t X(t) + AX(t) = f(t), \quad 0 < t < T, \quad X(0) = X_0$$



Domain Decomposition: Description of the method

Given $X_h^0, X_h^1, \dots, X_h^{n-1} \in S_h^0(\mathcal{D})$.

- Compute boundary conditions on each subdomain (from the previous iterates)
- Compute new solution $X_{h,i}^n$ on each subdomain $\mathcal{D}_i$
- Assemble the global solution

Method converges with rate $O(\tau^2)$

Main tools in the proof

- Boundary error: Exponential decay in the interior of the subdomain
- Induction: splitting of the error
  1. Stability of discrete solutions
  2. Convergence properties of Euler scheme
  3. Estimates for extrapolation

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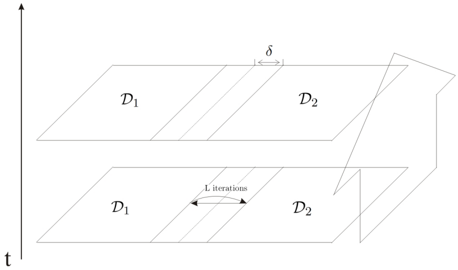
Advantage of Scheme A: stable, convergent, less computational effort, parallelizable
Restrictions for Stochastic Problems:

- Non-optimal weak convergence since Wiener process not differentiable (need of $\partial_{tt} X(t) \in L^2([0, T] \times \mathcal{D})$ in the analyis for the deterministic problem)

Problem setting for Scheme B

- Schwarz iteration: Iterative strategy to solve elliptic BVP.



- Motivation: parabolic problem after time discretization.

Idea of the Schwarz iteration

- Example: $-\Delta u = f \quad \text{on } \mathcal{D}, \quad u = 0 \quad \text{on } \partial\mathcal{D}$
- Iteration:

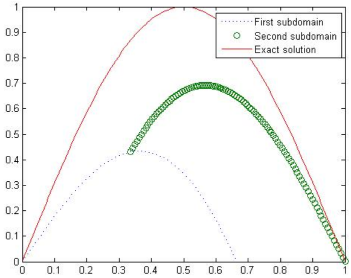
$$
\begin{aligned}
u_{h,0}^2 &= 0 \\
(\nabla u_{h,l}^1, \nabla v_h) &= (f, v_h) \quad \text{on } \mathcal{D}_1^\delta, \\
u_{h,l}^1 &= 0 \text{ on } \partial\mathcal{D}_1^\delta \cap \partial\mathcal{D}, \\
u_{h,l}^1 &= u_{l-1,h}^2 \text{ on } \partial\mathcal{D}_1^\delta \cap \partial\mathcal{D}_2^\delta
\end{aligned}
$$

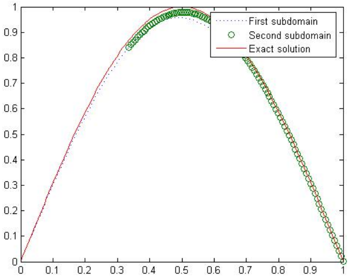
$$
\begin{aligned}
(\nabla u_{h,l}^2, \nabla v_h) &= (f, v_h) \quad \text{on } \mathcal{D}_2^\delta, \\
u_{h,l}^2 &= 0 \text{ on } \partial\mathcal{D}_2^\delta \cap \partial\mathcal{D}, \\
u_{h,l}^2 &= u_{l,h}^1 \text{ on } \partial\mathcal{D}_2^\delta \cap \partial\mathcal{D}_1^\delta
\end{aligned}
$$

From this iteration we get a global solution u_h^l on $\mathcal{D}$.

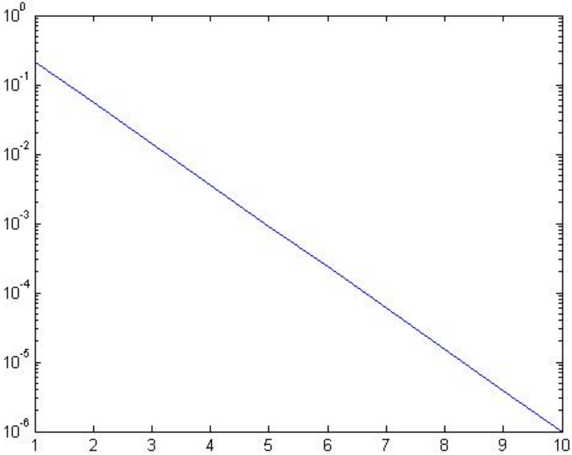
Solution of the Schwarz iteration

- $\mathcal{D} = (0,1)$, $\mathcal{D}_1^\delta = (0,2/3)$, $\mathcal{D}_1^\delta = (1/3,1)$, $\delta = 1/6$.





Error w.r.t number of iterations



Sketch of the proof

- Representation of the error $e_l = u_h - u_h^l$

$$e_l = Ee_{l-1} := (I - P_1)(I - P_2)e_{l-1}$$

  where $P_i : S_h^0(\mathcal{D}) \to S_h^0(\mathcal{D}_i^\delta)$ is the Ritz-projection
- Show that the operator E is a contraction
- Generalization for more subdomains: Bramble, Pasciak, Wang, Xu (1991).

Scheme A

Given $X_h^0 \in S_h^0(\mathcal{D})$. Let $n \geq 1$.

1. Compute new boundary conditions $X_{h,*}^n = \mathcal{E}(\{X_h^\mu\}_{\mu<n})$.
2. Find solution on each subdomain

$$(X_{h,i}^n - X_{h,i}^{n-1}, v_h) + \tau(\nabla X_{h,i}^n, \nabla v_h) = (Q^{1/2}\Delta W_{t_n}, v_h) \text{ on } \mathcal{D}_i^\delta$$
$$X_{h,i}^n = X_{h,i,*}^n \text{ on } \partial\mathcal{D}_i^\delta$$

3. Assemble the global solution $X_h^n = \mathcal{C}(\{X_{h,i}^n\}_i) \in S_h^0(\mathcal{D})$. Set $n \to n+1$.

Main result I

1. Overlap $\delta = C_0 h$,
2. $X \in L^2(\Omega; H^1(\mathcal{D}))$

Then solutions X^n of Algorithm A satisfy

$$\max_{0 \leq n \leq N} \left(\mathbb{E} \| X^n - X_{t_n} \|_{L^2}^2 \right)^{1/2} \leq C(\tau^{1/2} + h).$$

We couldn't obtain better results for weak convergence.

Main tools in the proof

- Boundary error (between solution of Euler step with global and local BC)
-

$$
\begin{aligned}
E^n := \mathbb{E}\|X_h^n - Y^n\|_{L^2}^2 &\leq C_1 \sum_{i=i}^{M} \mathbb{E}\|X_*^n - \tilde{Y}^n\|_{L^2(\mathcal{D}_i^\delta)} \\
&\quad + C_2 \mathbb{E}\|Y^n - \tilde{Y}^n\|_{L^2}^2 \\
&\leq C_3 \mathbb{E}\|\mathcal{E}\{X_h^n - Y^n\}\|_{L^2}^2 \\
&\quad + C_4 \mathbb{E}\|\mathcal{E}\{Y^n\} - Y^n\|_{L^2}^2 \\
&\quad + C_5 \mathbb{E}\|Y^n - \tilde{Y}^n\|_{L^2}^2
\end{aligned}
$$

Estimation: $red = E^{n-1}$, $blue \leq C\tau$, $green \leq CE^{n-1}$

Scheme B

- MOTIVATION: Obtain optimal rate of weak convergence.
- Given $X_h^0 \in S_h^0(\mathcal{D})$. Let $n \geq 1$.
  1. Perform L iterations on the problem

     $$(X_h^n - X_h^{n-1}, v_h) + \tau(\nabla X_h^n, \nabla v_h) = (\sqrt{\tau} Q^{1/2} \chi^n, v_h) \text{ on } \mathcal{D}$$

     to obtain local solutions $X_{h,i,L}^n$.
  2. Assemble the global solution $X_h^n = \mathcal{C}(\{X_{h,i,L}^n\}_i) \in S_h^0(\mathcal{D})$. Set $n \to n+1$ until $n = N$.

Main result II

1. L: number of iterations
2. $X \in L^2(\Omega; C([0,T]; L^2(\Omega)))$
3. $\left(1 - \frac{C_\delta^2}{C_0}\right)^{L/2} \leq \tau^{\alpha_L}$

Then for the solution X^n of Algorithm-B holds

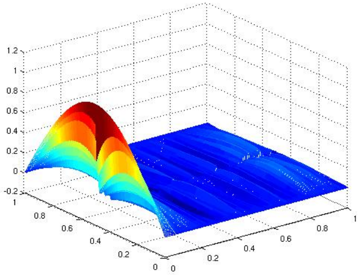
$$\max_{0\leq n\leq N} \sqrt{\mathbb{E}\|X^n - X_{t_n}\|_{L^2(\Omega;L^2(\mathcal{D}))}} \leq C(C_1\tau^{\alpha_L - 1} + \tau^{1/2} + h).$$

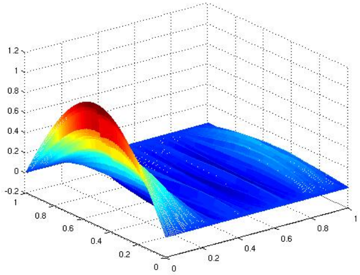
If moreover $\phi \in C_b^2$, then we have

$$\max_{0\leq n\leq N} \|\mathbb{E}\left[\phi(X^n) - \phi(X_{t_n})\right]\| \leq C(C_1\tau^{\alpha_L - 1} + \tau^{\gamma_w} + h^{2\gamma_w}).$$

Convergence results

- Conclusion: balance DD-error ($\tau^{\alpha_L - 1}$) with discretization error (τ^{γ_w}).





Dependence on the number of iterations

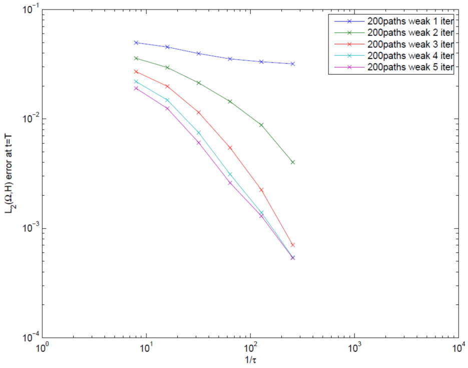
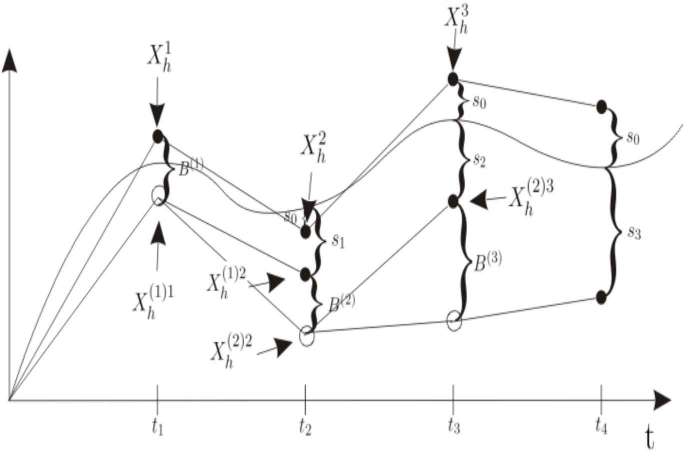


Figure: Rate of weak convergence

Accumulation of error perturbation in time

- Idea of the proof:



Sketch of the proof

$X_h^{(r)s}$, $r \leq s$ solution computed by Algorithm-B until time-step r, the BE until time-step s.
We only have to show

$$\max_{0 \leq n \leq N} \mathbb{E}\|X^{(n)n_h} - X_h^n\|_{L^2}^2 \leq C\tau^{\alpha_L - 1}$$

Recall that

$$\left(1 - \frac{C_\delta^2}{C_0}\right)^{L/2} \leq \tau^{\alpha_L}$$

- $s_n := \mathbb{E}\|X_h^{(n)k} - X_h^k\|_{L^2}^2 \leq Cn\tau^{\alpha_L}\exp(n\tau^{\alpha_L})$
- $B^{(n)} := \mathbb{E}\|X_h^{(n)k} - X_h^k\|_{L^2}^2 \leq C\tau^{\alpha_L}$

Summary

- Scheme A
  - Optimal strong convergence
  - Suboptimal weak convergence
- Scheme B
  - Optimal strong and weak convergence

Thank You for Your attention.