Weak order for the discretization of the stochastic heat equation driven by impulsive noise.

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Outline

Debussche/Printems: *Weak order for the discretization of the stochastic heat equation*, 2008.

$$dX_t + AX_t\,dt = Q^{1/2}dW_t,\ X_0 = x_0 \in H,\ t \in [0,T]. \tag{1}$$

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- Setting:
  $(X_t)_{t\in[0,T]}$ – H-valued stochastic process.
  A – positive self-adjoint unbounded operator on H with domain $D(A)$ dense in H and compactly embedded in H.
  Q – non negative symmetric bounded operator on H.
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- $(S(t))_{t\geq 0}$ – semigroup generated by A.
  The weak solution (in the PDE sense) of equation (1) is given by

$$X_t = S(t)x_0 + \int_0^t S(t-s)Q^{1/2}\,dW_s,\ t \in [0,T].$$

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- Main assumption concerning the spaces V_h: $\forall q \in [0,2]\ \exists\, \kappa_1, \kappa_2 > 0$,

$$\begin{aligned} \left\| S_h(t) P_h - S(t) \right\|_{L(H)} &\leq \kappa_1 h^q t^{-q/2}, \\ \left\| A^{1/2} \big(S_h(t) P_h - S(t) \big) \right\|_{L(H)} &\leq \kappa_2 h t^{-1}, \quad \forall h, t \geq 0. \end{aligned}$$

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$$\left\langle X_h^{n+1} - X_h^n, v_h \right\rangle_H + \Delta t \left\langle A\left(\theta X_h^{n+1} + (1-\theta) X_h^n\right), v_h \right\rangle_H = \left\langle Q^{1/2} Z_{t_{n+1}} - Q^{1/2} Z_{t_n}, v_h \right\rangle_H, \qquad \forall v_h \in V_h,$$

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$\Longrightarrow$

$$X_h^n = S_{h,\Delta t}^n P_h x_0 + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h Q^{1/2} (W_{t_{k+1}} - W_{t_k}).$$

$$\left(X_t = S(t) x_0 + \int_0^t S(t-s) Q^{1/2} \, dW_s \right)$$

Theorem (Debussche/Printems, 2008)

Assume that there exist real numbers $\alpha > 0$, $\beta \leq \alpha$ such that $\alpha - \beta \leq 1$ and

$$\begin{aligned} Tr(A^{-\alpha}) &< \infty\,, \\ A^{\beta}Q &\in L(H). \end{aligned}$$

Let $T \geq 1$, $\Delta t = T/N \leq 1$ and $\varphi \in C_b^2(H)$. Then there exists a constant $C = C(T,\varphi)$ which does not depend on h and N such that for any $\gamma < 1 - \alpha + \beta \leq 1$, the following inequality holds.

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Instead of $(W_t)_{t\in[0,T]}$, consider an impulsive cylindrical Process $(Z_t)_{t\in[0,T]}$.

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- Identify $\big(Z(t,\cdot)\big)_{t\in[0,T]}$ with the corresponding U-valued Lévy process $(Z_t)_{t\in[0,T]}$,

$$Z_t = L^2(\Omega, \mathcal{A}, \mathbb{P}; U) - \lim_{N\to\infty} \sum_{k=1}^N Z(t, e_k) e_k \qquad \big((e_k)_k \text{ ONB of } H\big).$$

  Impulsive cylindrical process on $L^2(\mathcal{O})$ with jump size intensity ν.

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- Analogously, for $\Phi \in L^2\big([0,T] \times \Omega, \mathcal{P}_{[0,T]}, dt\, d\mathbb{P};\, L_2(H)\big)$, $t \in [0,T]$,

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Theorem

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Remark:

An analogous result holds for equations of the form

$$dX_t + AX_t \, dt = Q_0^{1/2} dW_t + Q_1^{1/2} \, dZ_t + Q_2^{1/2} \, d\widetilde{Z}_t, \quad X_0 = x_0 \in H, \quad t \in [0, T].$$

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splitting of the error:

$$\begin{aligned}\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_h^N) &= \left\{\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T})\right\} + \left\{\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_h^N)\right\} \\ &= \text{spatial error} + \text{time discretization error}\end{aligned}$$

Estimate of the spatial error $\mathbb{E}\varphi(\mathbf{X_T}) - \mathbb{E}\varphi(\mathbf{X_{h,T}})$:

Define

$$v(t,x) \quad := \quad \mathbb{E}\varphi\left(x + \int_{T-t}^{T} S(T-r)Q^{1/2}dZ_r\right),\ t \in [0,T],\ x \in H,$$

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$$Y_t := S_h(T)P_h x_0 + \int_0^t S_h(T-r)P_h Q^{1/2}\, dZ_r,\ t\in[0,T],$$

then

$$\mathbb{E}\varphi(X_T) = v(T,S(T)x_0), \qquad \mathbb{E}\varphi(X_{h,T}) = \mathbb{E}v(0,X_{h,T}) = \mathbb{E}v(0,Y_T).$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T})$$
$$= \quad v(T,S(T)x_0) - \mathbb{E}v(0,Y_T)$$

Estimate of the spatial error $\mathbb{E}\varphi(\mathbf{X_T}) - \mathbb{E}\varphi(\mathbf{X_{h,T}})$:

Define

$$\begin{aligned} v(t,x) &:= \mathbb{E}\varphi\left(x + \int_{T-t}^{T} S(T-r)Q^{1/2}dZ_r\right), \; t \in [0,T], \; x \in H, \\ Y_t &:= S_h(T)P_h x_0 + \int_0^t S_h(T-r)P_h Q^{1/2}\, dZ_r, \; t \in [0,T], \end{aligned}$$

then

$$\mathbb{E}\varphi(X_T) = v(T, S(T)x_0), \qquad \mathbb{E}\varphi(X_{h,T}) = \mathbb{E}v(0, X_{h,T}) = \mathbb{E}v(0, Y_T).$$

$$\begin{aligned} &\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) \\ &\quad = \quad v(T,S(T)x_0) - \mathbb{E}v(0,Y_T) \\ &\quad \overset{\text{Itô}}{=} \quad v(T,S(T)x_0) - \mathbb{E}\Big[v(T,Y_0) + \int_0^T \big\langle D_x v(T-t,Y_{t-}),\, dZ_t\big\rangle_H \\ &\qquad - \int_0^T \frac{\partial v}{\partial t}(T-t,Y_{t-})\,dt + \frac{1}{2}\int_0^T \big\langle D_x^2 v(T-t,Y_{t-}),\, d[\![Z]\!]_t^c\big\rangle_{H\hat{\otimes}_2 H} \\ &\qquad + \sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \big\langle D_x v(T-t,Y_{t-}),\Delta Y_t\big\rangle_H\Big\}\Big] \end{aligned}$$

Estimate of the spatial error $\mathbb{E}\varphi(\mathbf{X_T}) - \mathbb{E}\varphi(\mathbf{X_{h,T}})$:

Define

$$v(t,x) := \mathbb{E}\varphi\left(x + \int_{T-t}^{T} S(T-r)Q^{1/2}dZ_r\right),\ t\in[0,T],\ x\in H,$$

$$Y_t := S_h(T)P_h x_0 + \int_0^t S_h(T-r)P_h Q^{1/2}\,dZ_r,\ t\in[0,T],$$

then

$$\mathbb{E}\varphi(X_T) = v(T, S(T)x_0), \qquad \mathbb{E}\varphi(X_{h,T}) = \mathbb{E}v(0, X_{h,T}) = \mathbb{E}v(0, Y_T).$$

$$\begin{aligned}
&\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T})\\
&= v(T,S(T)x_0) - \mathbb{E}v(0,Y_T)\\
&\overset{\text{Itô}}{=} v(T,S(T)x_0) - v(T,S_h(T)P_h x_0) \quad - \quad 0\\
&\quad + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \quad - \quad 0\\
&\quad - \mathbb{E}\sum_{t\le T}\left\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\right\}
\end{aligned}$$

Estimate of the spatial error $\mathbb{E}\varphi(\mathbf{X_T}) - \mathbb{E}\varphi(\mathbf{X_{h,T}})$:

Define

$$\begin{aligned} v(t,x) &:= \mathbb{E}\varphi\left(x + \int_{T-t}^{T} S(T-r)Q^{1/2}dZ_r\right),\ t \in [0,T],\ x \in H, \\ Y_t &:= S_h(T)P_h x_0 + \int_0^t S_h(T-r)P_h Q^{1/2}\, dZ_r,\ t \in [0,T], \end{aligned}$$

then

$$\mathbb{E}\varphi(X_T) = v(T, S(T)x_0), \qquad \mathbb{E}\varphi(X_{h,T}) = \mathbb{E}v(0, X_{h,T}) = \mathbb{E}v(0, Y_T).$$

$$\begin{aligned} &\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) \\ &= v(T, S(T)x_0) - \mathbb{E}v(0, Y_T) \\ &\overset{\text{Itô}}{=} v(T, S(T)x_0) - v(T, S_h(T)P_h x_0) \\ &\quad + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\, dt \\ &\quad - \mathbb{E}\sum_{t \le T}\left\{ v(T-t, Y_t) - v(T-t, Y_{t-}) - \langle D_x v(T-t, Y_{t-}), \Delta Y_t\rangle_H \right\} \end{aligned}$$

$$\begin{aligned} \mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\ &\quad - \mathbb{E}\sum_{t\le T}\Big\{ v(T-t, Y_t) - v(T-t, Y_{t-}) - \langle D_x v(T-t, Y_{t-}), \Delta Y_t\rangle_H \Big\}, \\ v(t,x) &= \mathbb{E}\varphi\left(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r \right). \end{aligned}$$

What does $\frac{\partial}{\partial t} v(t,x)$ look like?

$$\begin{aligned} \mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\ &\quad - \mathbb{E}\sum_{t\leq T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\}, \\ v(t,x) &= \mathbb{E}\varphi\left(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r\right). \end{aligned}$$

What does $\frac{\partial}{\partial t}v(t,x)$ look like?

$$\int_{T-t}^T S(T-r)Q^{1/2}dZ_r \sim \int_0^t S(r)Q^{1/2}dZ_r =: M_t.$$

$$\begin{aligned}\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\ &\quad - \mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\}, \\ v(t,x) &= \mathbb{E}\varphi\Big(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r\Big).\end{aligned}$$

What does $\frac{\partial}{\partial t}v(t,x)$ look like?

$$\int_{T-t}^T S(T-r)Q^{1/2}dZ_r \sim \int_0^t S(r)Q^{1/2}dZ_r =: M_t.$$

Applying Itô's formula,

$$\begin{aligned}\varphi(x+M_t) &= \varphi(x) + \text{martingale} + 0 \\ &\quad + \sum_{s\le t}\Big\{\varphi(x+M_{s-}+\Delta M_s) - \varphi(x+M_{s-}) - \big\langle D\varphi(x+M_{s-}), \Delta M_s\big\rangle_H\Big\}.\end{aligned}$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) = \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt$$
$$- \mathbb{E}\sum_{t\le T}\left\{v(T-t, Y_t) - v(T-t, Y_{t-}) - \langle D_x v(T-t, Y_{t-}), \Delta Y_t\rangle_H\right\},$$
$$v(t,x) = \mathbb{E}\varphi\left(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r\right).$$

What does $\frac{\partial}{\partial t}v(t,x)$ look like?

$$\int_{T-t}^T S(T-r)Q^{1/2}dZ_r \sim \int_0^t S(r)Q^{1/2}dZ_r =: M_t.$$

Applying Itô's formula,

$$\varphi(x+M_t) = \varphi(x) + \text{martingale}$$
$$+ \int_0^t \int_H \left\{\varphi(x+M_{s-}+y) - \varphi(x+M_{s-}) - \big\langle D\varphi(x+M_{s-}), y\big\rangle_H\right\}\mu_M(ds,dy).$$

$$\begin{aligned}\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\ &\quad - \mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\}, \\ v(t,x) &= \mathbb{E}\varphi\left(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r\right).\end{aligned}$$

What does $\frac{\partial}{\partial t} v(t,x)$ look like?

$$\int_{T-t}^T S(T-r)Q^{1/2}dZ_r \sim \int_0^t S(r)Q^{1/2}dZ_r =: M_t.$$

Applying Itô's formula,

$$\begin{aligned}\varphi\left(x+M_t\right) &= \varphi(x) + \text{martingale} \\ &\quad + \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} \Big\{\varphi(x + M_{s-} + \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k) - \varphi(x+M_{s-}) \\ &\qquad -\Big\langle D\varphi(x+M_{s-}), \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k\Big\rangle_H\Big\}\,\pi(ds, d\xi, d\sigma).\end{aligned}$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) = \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt$$
$$- \mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\},$$
$$v(t,x) = \mathbb{E}\varphi\Big(x + \int_{T-t}^T S(T-r)Q^{1/2}\,dZ_r\Big).$$

What does $\frac{\partial}{\partial t}v(t,x)$ look like?

$$\int_{T-t}^T S(T-r)Q^{1/2}dZ_r \sim \int_0^t S(r)Q^{1/2}dZ_r =: M_t.$$

Applying Itô's formula,

$$\varphi(x+M_t) = \varphi(x) + \text{martingale}$$
$$+ \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} \Big\{\varphi(x + M_{s-} + \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k) - \varphi(x+M_{s-})$$
$$- \Big\langle D\varphi(x+M_{s-}), \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k\Big\rangle_H\Big\}\,\pi(ds, d\xi, d\sigma).$$

Taking $\mathbb{E}\ldots$ and $\frac{\partial}{\partial t}\ldots$ yields

$$\frac{\partial}{\partial t}v(t,x) = \int_{\mathcal{O}}\int_{\mathbb{R}}\Big\{v\big(t, x + \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k\big) - v(t,x)$$
$$- \Big\langle Dv(t,x), \sum_k e_k(\xi)\sigma S(s)Q^{1/2}e_k\Big\rangle_H\Big\}\,d\xi\,\nu(d\sigma).$$

$$\begin{aligned}
\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\
&\quad - \mathbb{E}\sum_{t\leq T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\}, \\
Y_t &= S_h(T)x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\,dZ_s.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) &= \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt \\
&\quad - \mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\}, \\
Y_t &= S_h(T)x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\,dZ_s.
\end{aligned}
$$

Similarly, one gets

$$
\begin{aligned}
&\mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) + \langle D_x v(T-t,Y_{t-}),\, \Delta Y_t\rangle_H\Big\} \\
&= \mathbb{E}\int_0^T\int_H \Big\{v(T-t,Y_{t-}+y) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}),\, y\rangle_H\Big\}\mu_Y(dt,dy).
\end{aligned}
$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) = \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt$$
$$-\mathbb{E}\sum_{t\le T}\left\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\right\},$$
$$Y_t = S_h(T)x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\,dZ_s.$$

Similarly, one gets

$$\mathbb{E}\sum_{t\le T}\left\{v(T-t,Y_t) - v(T-t,Y_{t-}) + \langle D_x v(T-t,Y_{t-}),\, \Delta Y_t\rangle_H\right\}$$
$$= \mathbb{E}\int_0^T\int_H \left\{v(T-t,Y_{t-}+y) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}),\, y\rangle_H\right\}\mu_Y(dt,dy).$$
$$= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}} \Big\{v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) - v(T-t,Y_{t-})$$
$$-\Big\langle D_x v(T-t,Y_{t-}),\, \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big\rangle_H\Big\}\,\pi(dt,d\xi,d\sigma).$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) = \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt$$
$$- \mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\},$$
$$Y_t = S_h(T)x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\,dZ_s.$$

Similarly, one gets

$$\mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) + \langle D_x v(T-t,Y_{t-}),\, \Delta Y_t\rangle_H\Big\}$$
$$= \mathbb{E}\int_0^T\int_H \Big\{v(T-t,Y_{t-}+y) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}),\, y\rangle_H\Big\}\,\mu_Y(dt,dy).$$
$$= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}}\Big\{v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2}e_k\Big) - v(T-t,Y_{t-})$$
$$-\Big\langle D_x v(T-t,Y_{t-}),\, \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2}e_k\Big\rangle_H\Big\}\,dt\,d\xi\,\nu(d\sigma).$$

$$\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_{h,T}) = \ldots + \mathbb{E}\int_0^T \frac{\partial v}{\partial t}(T-t, Y_{t-})\,dt$$
$$-\mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}), \Delta Y_t\rangle_H\Big\},$$
$$Y_t = S_h(T)x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\,dZ_s.$$

Similarly, one gets

$$\mathbb{E}\sum_{t\le T}\Big\{v(T-t,Y_t) - v(T-t,Y_{t-}) + \langle D_x v(T-t,Y_{t-}),\, \Delta Y_t\rangle_H\Big\}$$
$$= \mathbb{E}\int_0^T\int_H \Big\{v(T-t,Y_{t-}+y) - v(T-t,Y_{t-}) - \langle D_x v(T-t,Y_{t-}),\, y\rangle_H\Big\}\,\mu_Y(dt,dy).$$
$$= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}} \Big\{v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2}e_k\Big) - v(T-t,Y_{t-})$$
$$-\Big\langle D_x v(T-t,Y_{t-}),\, \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2}e_k\Big\rangle_H\Big\}\,dt\,d\xi\,\nu(d\sigma).$$

All in all,

$$\begin{aligned}
&\left|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\right| \\
&\leq \quad Ch^{2\gamma} \\
&\quad + \Bigg|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2} e_k \Big\rangle_H \Big\} \, dt \, d\xi \, \nu(d\sigma)\Bigg|
\end{aligned}$$

All in all,

$$
\begin{aligned}
&\Big|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\Big| \\
&\le \quad Ch^{2\gamma} \\
&\quad + \Big|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad -v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2}e_k\Big\rangle_H \Big\} \,dt\, d\xi\, \nu(d\sigma)\Big| \\
&\le \quad Ch^{2\gamma} \\
&\quad + 2\|D_x v\|_{C_b(H)} \int \sigma\, \nu(d\sigma) \\
&\qquad \times \int_0^T \int_{\mathcal{O}} \Big|\sum_{k=1}^{\infty} e_k(\xi)\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2}e_k\Big|_H \,dt\, d\xi.
\end{aligned}
$$

All in all,

$$
\begin{aligned}
&\Big|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\Big| \\
&\quad\leq \quad Ch^{2\gamma} \\
&\qquad + \Bigg|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2}e_k\Big\rangle_H \Big\} \, dt\, d\xi\, \nu(d\sigma)\Bigg| \\
&\quad\leq \quad Ch^{2\gamma} \\
&\qquad + 2\|D_x v\|_{C_b(H)} \int \sigma\,\nu(d\sigma) \int_0^T \Big\|\Big(S(T-t) - S_h(T-t)P_h\Big)A^{(1-\gamma_1)/2}\Big\|_{L(H)}\, dt \\
&\qquad\qquad \times \int_{\mathcal{O}} \Big|\sum_{k=1}^\infty e_k(\xi) A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2} e_k\Big|_H \, d\xi.
\end{aligned}
$$

All in all,

$$
\begin{aligned}
&\left|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\right| \\
&\quad\le\quad Ch^{2\gamma} \\
&\qquad + \Bigg|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2} e_k\Big\rangle_H \Big\}\, dt\, d\xi\, \nu(d\sigma)\Bigg| \\
&\quad\le\quad Ch^{2\gamma} \\
&\qquad + 2\|D_x v\|_{C_b(H)} \int \sigma\, \nu(d\sigma) \int_0^T \left\|\Big(S(T-t) - S_h(T-t)P_h\Big)A^{(1-\gamma_1)/2}\right\|_{L(H)} dt \\
&\qquad\qquad \times |\mathcal{O}|^{1/2} \left\|A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2}\right\|_{L_2(H)}.
\end{aligned}
$$

All in all,

$$
\begin{aligned}
&\left|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\right| \\
&\leq \; Ch^{2\gamma} \\
&\quad + \Bigg|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2} e_k\Big\rangle_H \Big\}\, dt\, d\xi\, \nu(d\sigma)\Bigg| \\
&\leq \; Ch^{2\gamma} \\
&\quad + 2\|D_x v\|_{C_b(H)} \int \sigma\, \nu(d\sigma) \int_0^T \left\| \Big(S(T-t) - S_h(T-t)P_h\Big) A^{(1-\gamma_1)/2} \right\|_{L(H)} dt \\
&\qquad \times |\mathcal{O}|^{1/2} \left\| A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2} \right\|_{L_2(H)}.
\end{aligned}
$$

Finally,

$$
\left\| \Big(S(T-t) - S_h(T-t)P_h\Big) A^{(1-\gamma_1)/2} \right\|_{L(H)} \leq C\, h^{2\gamma} \left(t^{-\left(\gamma_1(\gamma_1-1)/(2\gamma)+1\right)} + t^{-((1-\gamma_1)/2+\gamma)} \right).
$$

□

Consider the case $\int_{-1}^{1} |\sigma| \, \nu(d\sigma) = \infty, \ \int_{-1}^{1} |\sigma|^2 \, \nu(d\sigma) < \infty$ **:**

Consider the case $\int_{-1}^{1} |\sigma|\,\nu(d\sigma) = \infty,\ \int_{-1}^{1} |\sigma|^2\,\nu(d\sigma) < \infty$:

$$\Big|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\Big|$$

$$\leq \quad Ch^{2\gamma}$$

$$+ \Big|\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\big)$$

$$-v\big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\big)$$

$$+ \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\big(S(T-t) - S_h(T-t)P_h\big)Q^{1/2} e_k\Big\rangle_H \Big\}\, dt\, d\xi\, \nu(d\sigma)\Big|$$

Consider the case $\int_{-1}^{1} |\sigma| \, \nu(d\sigma) = \infty$, $\int_{-1}^{1} |\sigma|^2 \, \nu(d\sigma) < \infty$:

$$
\begin{aligned}
&\left|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\right| \\
&\le \quad Ch^{2\gamma} \\
&\quad + \Bigg| \mathbb{E} \int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t)P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big)Q^{1/2} e_k \Big\rangle_H \Big\} \, dt \, d\xi \, \nu(d\sigma) \Bigg| \\
&\overset{\text{Taylor}}{=} \quad Ch^{2\gamma} \\
&\quad + \Bigg| \mathbb{E} \int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ \Big\langle \Sigma \, , \, D_x^2 v\Big(T-t, \, Y_{t-} + \vartheta\Sigma\Big)\Sigma \Big\rangle_H \\
&\qquad\qquad - \Big\langle \Sigma \, , \, D_x^2 v\Big(T-t, \, Y_{t-} + \vartheta\Sigma\Big)\Sigma \Big\rangle_H \Big\} \, dt \, d\xi \, \nu(d\sigma) \Bigg|
\end{aligned}
$$

Consider the case $\int_{-1}^{1} |\sigma| \, \nu(d\sigma) = \infty, \ \int_{-1}^{1} |\sigma|^2 \, \nu(d\sigma) < \infty$:

$$\begin{aligned}
&\Big|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\Big| \\
&\leq \quad Ch^{2\gamma} \\
&\quad + \Big|\mathbb{E} \int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S_h(T-t) P_h Q^{1/2} e_k\Big) \\
&\qquad\qquad - v\Big(T-t, Y_{t-} + \sum_k e_k(\xi)\sigma S(T-t) Q^{1/2} e_k\Big) \\
&\qquad + \Big\langle D_x v(T-t, Y_{t-}), \sum_k e_k(\xi)\sigma\Big(S(T-t) - S_h(T-t)P_h\Big) Q^{1/2} e_k \Big\rangle_H \Big\} \, dt \, d\xi \, \nu(d\sigma)\Big| \\
&\overset{\text{Taylor}}{=} \quad Ch^{2\gamma} \\
&\quad + \Big|\mathbb{E} \int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ \Big\langle \Sigma \,,\, D_x^2 v\big(T-t,\, Y_{t-} + \vartheta\Sigma\big) \Sigma \Big\rangle_H \\
&\qquad\qquad - \Big\langle \Sigma \,,\, D_x^2 v\big(T-t,\, Y_{t-} + \vartheta\Sigma\big) \Sigma \Big\rangle_H \Big\} \, dt \, d\xi \, \nu(d\sigma)\Big|
\end{aligned}$$

Besides other difficulties, this leads to expressions like

$$\int_{\mathcal{O}} \left| e_k(\xi) A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2} e_k \right|_H^3 d\xi.$$

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Thank you for your attention!